

Exact simultaneous recovery of locations and structure from known orientations and corrupted point correspondences.

Paul Hand*, Choongbum Lee and Vladislav Voroninski

Department of Mathematics, Massachusetts Institute of Technology

*Department of Computational and Applied Mathematics, Rice University

September 18, 2015

Abstract

Let $t_1, \dots, t_{n_l} \in \mathbb{R}^d$ and $p_1, \dots, p_{n_s} \in \mathbb{R}^d$ and consider the bipartite location recovery problem: given a subset of pairwise direction observations $\{(t_i - p_j)/\|t_i - p_j\|_2\}_{i,j \in [n_l] \times [n_s]}$, where a constant fraction of these observations are arbitrarily corrupted, find $\{t_i\}_{i \in [n_l]}$ and $\{p_j\}_{j \in [n_s]}$ up to a global translation and scale. We study the recently introduced ShapeFit algorithm as a method for solving this bipartite location recovery problem. In this case, ShapeFit consists of a simple convex program over $d(n_l + n_s)$ real variables. We prove that this program recovers a set of $n_l + n_s$ i.i.d. Gaussian locations exactly and with high probability if the observations are given by a bipartite Erdős-Rényi graph, d is large enough, and provided that at most a constant fraction of observations involving any particular location are adversarially corrupted. This recovery theorem is based on a set of deterministic conditions that we prove are sufficient for exact recovery. Finally, we propose a modified pipeline for the Structure for Motion problem, based on this bipartite location recovery problem.

1 Introduction

Structure from Motion (SfM) is the task of recovering 3d structure from a collection of images taken from different vantage points [6]. In the SfM problem, camera poses are represented by locations $t_i^{(0)} \in \mathbb{R}^3, i = 1 \dots n_\ell$ and rotation matrices $R_i \in SO(3), i = 1 \dots n_\ell$, where R_i maps coordinates in the frame of the i th camera to the world frame. For a generic structure point $p \in \mathbb{R}^3$, there exists a unique point in each imaging plane given by perspective projection. A pair of image points is said to correspond when they are both projections of the same point in 3d space. Given enough point correspondences between a pair of views, epipolar geometry yields the relative rotation and direction between those views. Pairwise relative camera poses can then be used to estimate the individual poses $(t_i^{(0)}, R_i), i = 1 \dots n_\ell$ up to a Euclidean transformation. Knowledge of camera poses and point correspondences allows one to estimate 3d structure via triangulation. Finally, the pose and structure estimates are used as initialization for bundle adjustment, which is the simultaneous nonlinear refinement of structure and camera poses. In summary, SfM typically consists of four steps: 1) identify point correspondences; 2) recover camera orientations and locations in global coordinates; 3) triangulate structure points using estimates of camera pose and correspondences; and 4) perform bundle adjustment.

A central difficulty of SfM is that point correspondences are prone to errors because they are found purely by local photometric information, which is subject to projective transformations from

camera motion, specularities, occlusions, variable lighting conditions, shadows, and repetitive structures commonly found in manmade scenes. Thus, every step of the above SfM pipeline needs to tolerate highly corrupted input data. For the correspondence step, techniques such as Random Sampling Consensus (RANSAC) are used to reduce the number of outliers among candidate correspondences initially obtained by brute-force photometric matching. Unfortunately, even after applying RANSAC, outliers in point correspondences are generally unavoidable.

Mathematically, once a set of correspondences has been established, the SfM problem can be formulated as the $d = 3$ case of the following. Let $T^{(0)}$ be a collection of n_ℓ distinct vectors $t_1^{(0)}, \dots, t_{n_\ell}^{(0)} \in \mathbb{R}^d$, and let $P^{(0)}$ be a collection of n_s distinct vectors $p_1^{(0)}, \dots, p_{n_s}^{(0)} \in \mathbb{R}^d$. Associated to locations $T^{(0)}$ is a set of orientations $R = \{R_i\}_{i \in [n_\ell]} \in SO(d)$. The pairs $(t_i^{(0)}, R_i)$ represents poses from which observations of the points $p_j^{(0)}$ are collected. Let $G(n_\ell, n_s, E)$ be a bipartite graph on $n_\ell + n_s$ vertices, where $E = E_g \sqcup E_b$, with E_b and E_g corresponding to pairwise direction observations that are respectively ‘corrupted’ and ‘uncorrupted.’ That is, for each $ij \in E$, we are given a vector v_{ij} , where

$$v_{ij} = \frac{R_i^t(t_i^{(0)} - p_j^{(0)})}{\|R_i^t(t_i^{(0)} - p_j^{(0)})\|_2} \text{ for } ij \in E_g, \quad v_{ij} \in \mathbb{S}^{d-1} \text{ for } ij \in E_b.$$

An uncorrupted observation v_{ij} is exactly the direction of $R_i^t(t_i^{(0)} - p_j^{(0)})$, and a corrupted observation is an arbitrary direction. Consider the task of finding the unknown locations $T^{(0)}$ and structure points $P^{(0)}$, up to a global translation and scale, and the orientations R , up to a global rotation, without knowledge of the decomposition $E = E_g \sqcup E_b$, nor the nature of the corruptions.

Estimating camera orientations R_i from from corrupted relative rotations $R_i^t R_j$ is a tractable and relatively well-understood problem. For instance, a method based on Lie group averaging performs well in practice [3], and a semidefinite program based on lifting and least unsquared deviations (LUD) has rigorous guarantees of exact recovery from corrupted relative rotations [11]. Once camera orientations are estimated, one can use epipolar geometry to obtain a set of relative direction estimates of camera locations. These estimates are partially corrupted since they are computed from the initial point correspondences. Camera locations in a global reference frame can be estimated using the 1dSfM approach of [12], which screens for outliers based on inconsistencies in 1d projections; however, this approach is not robust to self-consistent outliers. Alternatively, locations can be found by recent methods such as LUD [7] or the ShapeFit algorithm [4], which are both convex programs. It was proven in [4] that ShapeFit recovers locations exactly from partially corrupted pairwise directions under broad technical assumptions.

Having obtained an estimate of camera orientations and locations, one can recover an estimate of the 3d structure by triangulation, for instance by minimizing the quadratic reprojection error or maximizing a likelihood estimate. Bundle adjustment then proceeds by jointly optimizing this reprojection error or likelihood estimate with respect to camera poses and 3d structure. It is important to initialize bundle adjustment close to the global minimum, because it is non-convex and susceptible to getting stuck in local minima.

In this paper, we consider compressing two sub-steps of the pipeline — camera location recovery and structure recovery by triangulation — into one provably corruption-robust step based on the ShapeFit algorithm. Namely, once camera rotations are estimated, our approach uses the raw image coordinates of point correspondences to recover the camera locations and structure points simultaneously. If a structure point p_j is visible to a calibrated camera at location t_i , then its image coordinates under perspective projection provide a vector \tilde{v}_{ij} that has the same direction as $R_i^t(t_i - p_j)$. If the orientation R_i is known and accurate, then the direction of $t_i - p_j$ is also known.

Equivalently, if all the orientations R_i are known, we can take each R_i to be the identity without loss of generality. When a point correspondence is incorrect, the estimated direction of $t_i - p_j$ can of course be arbitrarily corrupted. We thus arrive at the following recovery problem.

With $T^{(0)}$ and $P^{(0)}$ defined as above, for each $ij \in E$, we are given a vector v_{ij} , where

$$v_{ij} = \frac{t_i^{(0)} - p_j^{(0)}}{\|t_i^{(0)} - p_j^{(0)}\|_2} \text{ for } ij \in E_g, \quad v_{ij} \in \mathbb{S}^{d-1} \text{ for } ij \in E_b. \quad (1)$$

Thus, an uncorrupted observation v_{ij} is exactly the direction of $t_i^{(0)} - p_j^{(0)}$, and a corrupted observation is an arbitrary direction. The task is to find the unknown locations $T^{(0)}$, $P^{(0)}$ up to global translation and scale, without knowledge of the decomposition $E = E_g \sqcup E_b$, nor the nature of the corruptions.

To summarize, we propose the following modified pipeline for Structure from Motion: 1) establish point correspondences; 2) estimate global orientations of the cameras; 3) estimate the camera locations and structure points simultaneously; and 4) run bundle adjustment.

We will show that ShapeFit, a tractable convex program, can exactly solve the recovery problem in Step 3 under broad deterministic assumptions and under a random model. In [4], the present authors showed that ShapeFit recovers camera locations exactly from corrupted pairwise direction under suitable assumptions. The result in [4] strongly relies on the existence of triangles in the graph of observations, whereas in our present setting, the underlying graphs are bipartite and necessarily do not contain triangles. In this bipartite setting, we will prove a deterministic recovery result for ShapeFit based on the presence of cycles of length 4. We also show that under a random Gaussian and Erdos-Renyi model, ShapeFit recovers structure and locations exactly from known orientations and corrupted correspondences with high probability in the high dimensional case. To the best of our knowledge, these are the first theoretical results guaranteeing exact location and structure recovery from corrupted correspondences and known orientations.

1.1 Problem formulation

The location recovery problem is to recover a set of points in \mathbb{R}^d from observations of pairwise directions between those points. Since relative direction observations are invariant under a global translation and scaling, one can at best hope to recover the locations $T^{(0)} = \{t_1^{(0)}, \dots, t_n^{(0)}\}$ and structure points $P^{(0)} = \{p_1^{(0)}, \dots, p_n^{(0)}\}$ up to such a transformation. That is, successful recovery from $\{v_{ij}\}_{(i,j) \in E}$ is finding two sets of vectors $\{\alpha(t_i^{(0)} + w)\}_{i \in [n_\ell]}, \{\alpha(p_j^{(0)} + w)\}_{j \in [n_s]}$ for some $w \in \mathbb{R}^d$ and $\alpha > 0$. We will say that two pairs of sets of vectors (T, P) and $(T^{(0)}, P^{(0)})$ are equal up to global translation and scale if there exists a vector w and a scalar $\alpha > 0$ such that $t_i = \alpha(t_i^{(0)} + w)$ for all $i \in [n_\ell]$ and $p_j = \alpha(p_j^{(0)} + w)$ for all $j \in [n_s]$. In this case, we will say that (T, P) and $(T^{(0)}, P^{(0)})$ have the same ‘shape,’ and we will denote this property as $(T, P) \sim (T^{(0)}, P^{(0)})$. The location recovery problem is then stated as:

Given: $G(n_\ell, n_s, E)$, $\{v_{ij}\}_{ij \in E}$ satisfying (1)

Find: $T = \{t_1, \dots, t_{n_\ell}\} \in \mathbb{R}^{d \times n_\ell}, P = \{p_1, \dots, p_{n_s}\} \in \mathbb{R}^{d \times n_s}$ such that $(T, P) \sim (T^{(0)}, P^{(0)})$

For this problem to be information theoretically well-posed under arbitrary corruptions, the maximum number of corrupted observations affecting any particular location t_i must be at most $\frac{n_s}{2}$. Similarly, the maximum number affecting any particular structure point p_j must be at most $\frac{n_\ell}{2}$. Otherwise, suppose that for some location $t_i^{(0)}$ of structure point $p_j^{(0)}$, half of its associated

observations v_{ij} are consistent with $t_i^{(0)}$, and the other half are corrupted so as to be consistent with some arbitrary alternative location w . Distinguishing between $t_i^{(0)}$ and w is then impossible in general. A similar argument follows for some structure point $p_j^{(0)}$. Formally, let $\deg_b(t_i)$ be the degree of location t_i in the graph $G(n_\ell, n_s, E_b)$ and let $\deg_b(p_j)$ be the degree of structure point p_j in the graph $G(n_\ell, n_s, E_b)$. Then, well-posedness under adversarial corruption requires that $\max_{i \in [n_\ell]} \deg_b(t_i) \leq \gamma n_\ell$ and $\max_{j \in [n_s]} \deg_b(p_j) \leq \gamma n_s$, for some $\gamma < 1/2$,

Beyond the above necessary degree condition on E_g for well-posedness of recovery, we do not assume anything about the nature of corruptions. That is, we work with adversarially chosen corrupted edges E_b and arbitrary corruptions of observations associated to those edges. To solve the location recovery problem in this challenging setting, we utilize the convex program called ShapeFit [4]:

$$\min_{\substack{\{t_i\}_{i \in [n_\ell]} \\ \{p_j\}_{j \in [n_s]}}} \sum_{ij \in E} \|P_{v_{ij}^\perp}(t_i - p_j)\|_2 \quad \text{subject to} \quad \sum_{ij \in E} \langle t_i - p_j, v_{ij} \rangle = 1, \quad \sum_{i=1}^{n_\ell} t_i + \sum_{j=1}^{n_s} p_j = 0 \quad (2)$$

where $P_{v_{ij}^\perp}$ is the projector onto the orthogonal complement of the span of v_{ij} .

This convex program is a second order cone problem with $d(n_\ell + n_s)$ variables and two constraints. Hence, the search space has dimension $d(n_\ell + n_s) - 2$, which is minimal due to the $d(n_\ell + n_s)$ degrees of freedom in the locations $\{t_i\}$ and structure points $\{p_j\}$ and the two inherent degeneracies of translation and scale.

1.2 Main result

In this paper, we consider the model where the n_ℓ locations and n_s structure points are i.i.d. Gaussian, and where pairwise direction observations are given according to an Erdős-Rényi bipartite random graph. We show that in a high-dimensional setting, ShapeFit *exactly* recovers the locations and structure points with high probability, provided that n_ℓ and n_s are sub-exponential in d , and provided that at most a fixed fraction of observations are *adversarially* corrupted.

Theorem 1. *Let $N = \max(n_\ell, n_s)$, $n = \min(n_\ell, n_s)$. Let $G(V_\ell \cup V_s, E)$ be drawn from a bipartite Erdős-Rényi graph with $p > 0$. Take $t_1^{(0)}, \dots, t_{n_\ell}^{(0)}, p_1^{(0)}, \dots, p_{n_s}^{(0)} \sim \mathcal{N}(0, I_{d \times d})$ to be independent from each other and G . Then, there exist absolute constants $c, c_3, C > 0$ such that for $\gamma = c_3 p^4$, if*

$$\max\left(\frac{1}{c_3 p^4}, Cd, \frac{2 \log(eN)}{p}, \Omega(c_3 \log^2 N)\right) \leq n \leq N \leq e^{\frac{1}{8}cd}$$

and $d = \Omega(1)$, then there exists an event with probability at least $1 - O(e^{-\Omega(\frac{1}{2}c_3^{-1/2}n^{1/2})} + e^{-\frac{1}{2}cd})$, on which the following holds:

For all subgraphs E_b satisfying $\max_{i \in [n_\ell]} \deg_b(t_i) \leq \gamma n_s$ and $\max_{j \in [n_s]} \deg_b(p_j) \leq \gamma n_\ell$ and all pairwise direction corruptions $v_{ij} \in \mathbb{S}^{d-1}$ for $ij \in E_b$, the convex program (2) has a unique minimizer equal to $\left\{ \alpha \{t_i^{(0)} - \zeta\}_{i \in [n_\ell]}, \alpha \{p_j^{(0)} - \zeta\}_{j \in [n_s]} \right\}$ for some positive α and for $\zeta = \frac{1}{n_\ell + n_s} \left(\sum_{i \in [n_\ell]} t_i^{(0)} + \sum_{j \in [n_s]} p_j^{(0)} \right)$.

This probabilistic recovery theorem is based on a set of deterministic conditions that we prove are sufficient to guarantee exact recovery. These conditions are satisfied with high probability in the model described above. See Section 2.1 for the deterministic conditions.

This recovery theorem is high-dimensional in the sense that the probability estimate and the exponential upper bound on $n_\ell + n_s$ are only meaningful for $d = \Omega(1)$. Concentration of measure

in high dimensions and the upper bound on $n_\ell + n_s$ ensure control over the angles and distances between random points. As a result, lower dimensional spaces are a more challenging regime for recovery.

Numerical simulations empirically verify the main message of these recovery theorem: ShapeFit simultaneously recovers a set of locations and structure points exactly from corrupted direction observations, provided that up to a constant fraction of the observations at each location and structure point are corrupted. We present numerical studies in the physically relevant setting of \mathbb{R}^3 , with an underlying random Erdős-Rényi bipartite graph of observations. Further numerical simulations show that recovery is stable to the additional presence of noise on the uncorrupted measurements. That is, locations and structure points are simultaneously recovered approximately under such conditions, with a favorable dependence of the estimation error on the measurement noise.

1.3 Organization of the paper

Section 1.4 presents the notation used throughout the rest of the paper. Section 2 presents the proof of Theorem 1. Section 3 presents results from numerical simulations.

1.4 Notation

Let $[k] = \{1, \dots, k\}$. Let $V_\ell = [n_\ell]$ and $V_s = [n_s]$. Let $N = \max(n_\ell, n_s)$ and $n = \min(n_\ell, n_s)$. Let e_i be the i th standard basis element. For a bipartite graph $G(V_\ell \cup V_s, E)$, we write an arbitrary edge as an ordered pair (i, j) , where $i \in V_\ell$ and $j \in V_s$. Let K_{n_ℓ, n_s} be the complete bipartite graph on $n_\ell + n_s$ vertices. A cycle of length 4 will be denoted as C_4 . Let $E(K_{n_\ell, n_s})$ be the set of edges in K_{n_ℓ, n_s} . Let $\|\cdot\|_2$ be the standard ℓ_2 norm on a vector. For any nonzero vector v , let $\hat{v} = v/\|v\|_2$. For a subspace W , let P_W be the orthogonal projector onto W . For a vector v , let P_{v^\perp} be the orthogonal projector onto the orthogonal complement of the span of $\{v\}$.

Let T denote the set $T = \{t_i\}_{i \in V_\ell}$, for $t_i \in \mathbb{R}^d$. Let P denote the set $P = \{p_j\}_{j \in V_s}$, for $p_j \in \mathbb{R}^d$. For $i \in V_\ell, j \in V_s$, define $t_{ij} = t_i - p_j$ for all $i \in V_\ell, j \in V_s$. For $i, k \in V_\ell$, define $t_{ik} = t_i - t_k$. For $j, l \in V_s$, define $t_{jl} = p_j - p_l$. Define $\bar{\zeta} = \frac{1}{n_\ell + n_s} \left(\sum_{i \in V_\ell} t_i + \sum_{j \in V_s} p_j \right)$. Define $t_{ij}^{(0)}$, $T^{(0)}$, $\bar{\zeta}^{(0)}$, $P^{(0)}$, similarly. We define $\mu_\infty = \max_{i \neq j} \|t_{ij}^{(0)}\|_2$. For a scalar c and a set of vectors $X \subseteq \mathbb{R}^d$, let $cX = \{cx : x \in X\}$. For a given $G = G(V_\ell \cup V_s, E)$ and $\{v_{ij}\}_{ij \in E}$, where $v_{ij} \in \mathbb{R}^d$ have unit norm, let $R(T, P) = \sum_{ij \in E} \|P_{v_{ij}^\perp} t_{ij}\|_2$. Let $L(T, P) = \sum_{ij \in E} \langle t_{ij}, v_{ij} \rangle$. Let $\ell_{ij} = \langle t_{ij}, v_{ij} \rangle$, and similarly for $\ell_{ij}^{(0)}$. In this notation, ShapeFit is

$$\min_{T, P} R(T, P) \quad \text{subject to} \quad L(T, P) = 1, \quad \bar{\zeta} = 0$$

For vectors v_1, \dots, v_k , let $S(v_1, \dots, v_k) = \text{span}(v_1, \dots, v_k)$ be the vector space spanned by these vectors. Given t_{ij} and $t_{ij}^{(0)}$, define δ_{ij} , η_{ij} , and s_{ij} such that

$$t_{ij} = (1 + \delta_{ij})t_{ij}^{(0)} + \eta_{ij}s_{ij}$$

where s_{ij} is a unit vector orthogonal to $t_{ij}^{(0)}$ and $\eta_{ij} = \|P_{t_{ij}^{(0)\perp}} t_{ij}\|_2$. Note that $\eta_{ij} \geq 0$.

2 Proofs

We will prove Theorem 1 using the same general strategy as in [4]. Specifically, the proof of Theorem 1 can be separated into two parts: a recovery guarantee under a set of deterministic conditions,

and a proof that the random model meets these conditions with high probability. These sufficient deterministic conditions, roughly speaking, are (1) that the graph is connected and the nodes have tightly controlled degrees; (2) that the camera and structure locations are all distinct; (3) that all pairwise distances between cameras and locations are within a constant factor of each other; (4) that any choice of two camera locations and two structure locations live in a three dimensional affine space; (5) that the camera and structure locations are ‘well-distributed’ in a sense that we will make precise; and (6) that there are not too many corruptions affecting a single camera location or structure point. Theorem 2 in Section 2.1 states these deterministic conditions formally.

As in [4], we will prove the deterministic recovery theorem directly, using several geometric properties concerning how deformations of a set of points induce rotations. Note that an infinitesimal rigid rotation of two points $\{t_i, t_j\}$ about their midpoint to $\{t_i + h_i, t_j + h_j\}$ is such that $h_i - h_j$ is orthogonal to $t_{ij} = t_i - t_j$. We will abuse terminology and say that $\|P_{t_{ij}^\perp}(h_i - h_j)\|$ is a measure of the rotation in a finite deformation $\{h_i, h_j\}$, and we say that $\langle h_i - h_j, t_i - t_j \rangle$ is the amount of stretching in that deformation. Using this terminology, the geometric properties we establish are:

- If a deformation stretches two adjacent sides of a C_4 at different rates, then that induces a rotation in some edge of the C_4 (Lemma 2).
- If a deformation rotates one edge shared by many C_4 s, then it induces a rotation over many of those C_4 s, provided the opposite points of those triangles are ‘well-distributed’ (Lemma 3).
- A deformation that rotates bad edges, must also rotate good edges (Lemma 4).
- For any deformation, some fraction of the sum of all rotations must affect the good edges (Lemma 5).

By using these geometric properties, we show that all nonzero feasible deformations induce a large amount of total rotation. Since some fraction of the total rotation must be on the good edges, the objective must increase.

The main technical difference between the present proof and the proof of [4] is that the proof in [4] relies on the presence of many triangles in the graph of uncorrupted measurements. Because of the bipartite structure of the present work, there are no triangles in the graph. Hence, the technical novelty of the present proof is the establishment of the properties above when there are a sufficient number of C_4 s in the graph of uncorrupted measurements.

In Section 2.1, we present the deterministic recovery theorem. In Section 2.2, we present and prove Lemma 2. In Section 2.3, we present and prove Lemmas 3–5. In Section 2.4, we prove the deterministic recovery theorem. In Section 2.5, we prove that Gaussians satisfy several properties with high probability. In Section 2.6, we prove that Gaussians satisfy well-distributedness with high probability. In Section 2.7, we prove that Erdős-Rényi graphs are connected and have controlled degrees and codegrees with high probability. Finally, in Section 2.8, we prove Theorem 1.

2.1 Deterministic recovery theorem in high dimensions

To state the deterministic recovery theorem, we need two definitions. The first definition captures the ‘regularity’ of the measurement graph. A random bipartite graph can easily be seen to satisfy the conditions. Note that the definition does not depend on the vectors locations $\{t_i\}$ and $\{p_j\}$.

Definition 1. *We say that a graph $G(V_\ell \cup V_s, E)$ is bipartite- p -typical if it satisfies the following properties:*

1. G is connected,
2. each vertex in V_ℓ has degree between $\frac{1}{2}n_s p$ and $2n_s p$, and each vertex in V_s has degree between $\frac{1}{2}n_\ell p$ and $2n_\ell p$.
3. each pair of vertices in V_ℓ has codegree between $\frac{1}{2}n_s p^2$ and $2n_s p^2$, where the codegree of $j, l \in V_\ell = |\{i \mid ij \in E(G), il \in E(G)\}|$. Each pair of vertices in V_s has codegree between $\frac{1}{2}n_\ell p^2$ and $2n_\ell p^2$.

The next definition captures how ‘well-distributed’ the location points $\{t_i\}$ and $\{p_j\}$ are in \mathbb{R}^d .

Definition 2.

- (i) Let $S = \{(t_k, p_k)\}_{k=1\dots m} \subset \mathbb{R}^d \times \mathbb{R}^d$. Let $x, y \in \mathbb{R}^d$. We say that S is c -well-distributed with respect to (x, y) if the following holds for all $h \in \mathbb{R}^d$:

$$\sum_{(t,p) \in S} \|P_{\text{span}\{p-x, t-p, y-t\}^\perp}(h)\|_2 \geq c|S| \cdot \|P_{(x-y)^\perp}(h)\|_2.$$

- (ii) Let $T = \{t_i\}_{i \in V_\ell}$ and $P = \{p_j\}_{j \in V_s}$. We say that (T, P) is c -well-distributed along G if for all $i \in V_\ell, j \in V_s$, the set $S_{ij} = \{(t_k, p_\ell) : il \in E(G), k\ell \in E(G), kj \in E(G), k \neq i, \ell \neq j\}$ is c -well-distributed with respect to (t_i, p_j) .

We now state sufficient deterministic recovery conditions on the graph G , the subgraph E_b corresponding to corrupted observations, and the locations $T^{(0)}$ and $P^{(0)}$.

Theorem 2. Suppose $T^{(0)}, P^{(0)}, E_b, G$ satisfy the conditions

1. The underlying graph G is bipartite- p -typical,
2. All vectors in $T^{(0)}, P^{(0)}$ and $T^{(0)} \cup P^{(0)}$ are distinct, respectively.
3. For all $i, k \in V_\ell$ and $j, \ell \in V_s$, we have $c_0 \|t_{k\ell}^{(0)}\|_2 \leq \|t_{ij}^{(0)}\|_2$,
4. For all $i, k \in V_\ell, j, \ell \in V_s$ such that $k \neq i, j \neq \ell$, we have $\min\left(\|P_{\text{span}(t_{kj}^{(0)}, t_{i\ell}^{(0)})^\perp} t_{ij}^{(0)}\|_2, \|P_{\text{span}(t_{k\ell}^{(0)}, t_{ij}^{(0)})^\perp} t_{ij}^{(0)}\|_2\right) / \|t_{ij}^{(0)}\|_2 \geq \beta$
5. The pair $(T^{(0)}, P^{(0)})$ is c_1 -well-distributed along G ,
6. Each vertex in V_ℓ (resp. V_s) has at most εn_s (resp. εn_ℓ) incident edges in E_b .

for constants $0 < p, c_0, \beta, c_1, \varepsilon \leq 1$. If $\varepsilon \leq \frac{\beta c_0 c_1^2 p^4}{384 \cdot 204 \cdot 64}$ and $n_\ell, n_s > \max(64, \frac{8}{p^2})$, then $L(T^{(0)}, P^{(0)}) \neq 0$ and $(T^{(0)}, P^{(0)})/L(T^{(0)}, P^{(0)})$ is the unique optimizer of ShapeFit.

Before we prove the theorem, we establish that $L(T^{(0)}, P^{(0)}) \neq 0$ when ε is small enough. This property guarantees that some scaling of $(T^{(0)}, P^{(0)})$ is feasible and occurs, roughly speaking, when $|E_b| < |E_g|$.

Lemma 1. If $\varepsilon < \frac{c_0 p}{4}$, then $L(T^{(0)}, P^{(0)}) \neq 0$.

Proof. Since $v_{ij} = \hat{t}_{ij}^{(0)}$ for all $ij \in E_g$, we have

$$L(T^{(0)}, P^{(0)}) = \sum_{ij \in E(G)} \langle t_{ij}^{(0)}, v_{ij} \rangle \geq \sum_{ij \in E_g} \|t_{ij}^{(0)}\|_2 - \sum_{ij \in E_b} \|t_{ij}^{(0)}\|_2.$$

By Condition 3, $c_0 \mu_\infty |E_g| \leq \sum_{ij \in E_g} \|t_{ij}^{(0)}\|_2$ and $\mu_\infty |E_b| \geq \sum_{ij \in E_b} \|t_{ij}^{(0)}\|_2$. Thus it suffices to prove that $c_0 |E_g| > |E_b|$. As $\varepsilon < \frac{p}{4}$, Condition 1 and 6 gives $|E_g| \geq \frac{1}{2} n_\ell n_s p - \varepsilon n_\ell n_s \geq \frac{1}{4} n_\ell n_s p$. Since $|E_b| \leq \varepsilon n_\ell n_s$, if $\varepsilon < \frac{c_0 p}{4}$, then we have $c_0 |E_g| > |E_b|$. \square

2.2 Unbalanced parallel motions induce rotation

The following lemma concerns geometric properties of deformations of a set of points. Specifically it shows that if four points are deformed in a way that differentially scales the lengths of two edges, then it necessarily induces a rotation somewhere in a C_4 containing those points.

Lemma 2. *Let $d \geq 3$. Let $t_1, t_2, t_3, t_4 \in \mathbb{R}^d$ be distinct. Let $t_{ij} = t_i - t_j$ and $\hat{t}_{ij} = \frac{t_{ij}}{\|t_{ij}\|}$. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}$. Let $\{\tilde{\delta}_{i(i+1)}\}$ be such that $\langle v_i - v_{i+1} - \alpha t_{i(i+1)}, \hat{t}_{i(i+1)} \rangle = \tilde{\delta}_{i(i+1)} \|t_{i(i+1)}\|_2$ for each $i \in [4]$, where index summation is modulo 4.*

$$\begin{aligned} (i) \quad & \sum_{i \in [4]} \|P_{t_{i(i+1)}^\perp} (v_i - v_{i+1})\|_2 \geq \|P_{\text{span}(t_{23}, t_{41})^\perp} t_{12}\|_2 |\tilde{\delta}_{12} - \tilde{\delta}_{34}| \\ (ii) \quad & \sum_{i \in [4]} \|P_{t_{i(i+1)}^\perp} (v_i - v_{i+1})\|_2 \geq \|P_{\text{span}(t_{34}, t_{41})^\perp} t_{12}\|_2 |\tilde{\delta}_{12} - \tilde{\delta}_{23}|. \end{aligned}$$

Proof. The given condition implies $P_{t_{i(i+1)}^\perp} (v_i - v_{i+1}) = v_i - v_{i+1} - (\alpha + \tilde{\delta}_{i(i+1)}) t_{i(i+1)}$ for each $i \in [4]$. Therefore,

$$\begin{aligned} \sum_{i \in [4]} \|P_{t_{i(i+1)}^\perp} (v_i - v_{i+1})\|_2 &= \sum_{i \in [4]} \|v_i - v_{i+1} - (\alpha + \tilde{\delta}_{i(i+1)}) t_{i(i+1)}\|_2 \\ &\geq \left\| \sum_{i \in [4]} v_i - v_{i+1} - (\alpha + \tilde{\delta}_{i(i+1)}) t_{i(i+1)} \right\|_2 \\ &= \|\tilde{\delta}_{12} t_{12} + \tilde{\delta}_{23} t_{23} + \tilde{\delta}_{34} t_{34} + \tilde{\delta}_{41} t_{41}\|_2. \end{aligned} \quad (3)$$

(i) Since $\tilde{\delta}_{34}(t_{12} + t_{23} + t_{34} + t_{41}) = 0$, the right-hand-side of (3) equals $\|(\tilde{\delta}_{12} - \tilde{\delta}_{34})t_{12} + (\tilde{\delta}_{23} - \tilde{\delta}_{34})t_{23} + (\tilde{\delta}_{41} - \tilde{\delta}_{34})t_{41}\|_2$. The conclusion follows since

$$\begin{aligned} \|(\tilde{\delta}_{12} - \tilde{\delta}_{34})t_{12} + (\tilde{\delta}_{23} - \tilde{\delta}_{34})t_{23} + (\tilde{\delta}_{41} - \tilde{\delta}_{34})t_{41}\|_2 &\geq \min_{s, s' \in \mathbb{R}} \|(\tilde{\delta}_{12} - \tilde{\delta}_{34})t_{12} - st_{23} - s't_{41}\|_2 \\ &= \|P_{\text{span}(t_{23}, t_{41})^\perp} (\tilde{\delta}_{12} - \tilde{\delta}_{34})t_{12}\|_2. \end{aligned}$$

(ii) Since $\tilde{\delta}_{23}(t_{12} + t_{23} + t_{34} + t_{41}) = 0$, the right-hand-side of (3) equals $\|(\tilde{\delta}_{12} - \tilde{\delta}_{23})t_{12} + (\tilde{\delta}_{34} - \tilde{\delta}_{23})t_{34} + (\tilde{\delta}_{41} - \tilde{\delta}_{23})t_{41}\|_2$. The conclusion follows since

$$\begin{aligned} \|(\tilde{\delta}_{12} - \tilde{\delta}_{23})t_{12} + (\tilde{\delta}_{34} - \tilde{\delta}_{23})t_{34} + (\tilde{\delta}_{41} - \tilde{\delta}_{23})t_{41}\|_2 &\geq \min_{s, s' \in \mathbb{R}} \|(\tilde{\delta}_{12} - \tilde{\delta}_{23})t_{12} - st_{34} - s't_{41}\|_2 \\ &= \|P_{\text{span}(t_{34}, t_{41})^\perp} (\tilde{\delta}_{12} - \tilde{\delta}_{23})t_{12}\|_2. \quad \square \end{aligned}$$

2.3 C_4 s inequality and rotation propagation

The following lemma is a generalization of the triangle inequality in a context of the rotational part of structure deformations.

Lemma 3 (C_4 s Inequality). *Let $d \geq 4$; $x, y \in \mathbb{R}^d$. Let $S = \{(t_1, p_1), \dots, (t_k, p_k)\} \subset \mathbb{R}^d \times \mathbb{R}^d$. If S is c -well-distributed with respect to (x, y) , then for all vectors $h_x, h_y, h_{t_1}, \dots, h_{t_k}, h_{p_1}, \dots, h_{p_k} \in \mathbb{R}^d$ and sets $X \subseteq [k]$, we have*

$$\sum_{i \in [k] \setminus X} \|P_{(x-p_i)^\perp} (h_x - h_{p_i})\|_2 + \|P_{(p_i-t_i)^\perp} (h_{p_i} - h_{t_i})\|_2 + \|P_{(t_i-y)^\perp} (h_{t_i} - h_y)\|_2 \geq (ck - |X|) \cdot \|P_{(x-y)^\perp} (h_x - h_y)\|_2.$$

Proof. For each $i \in [k]$, define $W_i = \text{span}(x - p_i, p_i - t_i, t_i - y)$. Define P as the projection map to the space of vectors orthogonal to $x - y$, and define P_i for each $i \in [k]$ as the projection map to W_i^\perp . Since $(x - p_i)^\perp \supseteq W_i^\perp$, $(p_i - t_i)^\perp \supseteq W_i^\perp$, and $(t_i - y)^\perp \supseteq W_i^\perp$, it follows that

$$\begin{aligned} & \sum_{i \in [k] \setminus X} \|P_{(x-p_i)^\perp}(h_x - h_{p_i})\|_2 + \|P_{(p_i-t_i)^\perp}(h_{p_i} - h_{t_i})\|_2 + \|P_{(t_i-y)^\perp}(h_{t_i} - h_y)\|_2 \\ & \geq \sum_{i \in [k] \setminus X} \|P_i(h_x - h_{p_i})\|_2 + \|P_i(h_{p_i} - h_{t_i})\|_2 + \|P_i(h_{t_i} - h_y)\|_2 \geq \sum_{i \in [k] \setminus X} \|P_i(h_x - h_y)\|_2. \end{aligned}$$

Since $\{(t_1, p_1), \dots, (t_k, p_k)\}$ are well-distributed with respect to (x, y) , we have

$$\sum_{i \in [k]} \|P_i(h_x - h_y)\|_2 \geq ck \cdot \|P(h_x - h_y)\|_2. \quad (4)$$

Since $(x - y)^\perp \supseteq W_i^\perp$, we have $\|P_i(h_x - h_y)\|_2 \leq \|P(h_x - h_y)\|_2$ for all i . Hence

$$\sum_{i \in [k] \setminus X} \|P_i(h_x - h_y)\|_2 \geq (ck - |X|) \cdot \|P(h_x - h_y)\|_2,$$

proving the lemma. \square

The proof of Theorem 2 will rely on the following two lemmas, which state that rotational motions on some parts of the graph bound rotational motions on other parts. The following lemma relates the rotational motions on bad edges to the rotational motions on good edges. Recall the notation $t_{ij} = (1 + \delta_{ij})t_{ij}^{(0)} + \eta_{ij}s_{ij}$ where s_{ij} is a unit vector orthogonal to $t_{ij}^{(0)}$ and $\eta_{ij} = \|P_{t_{ij}^{(0)\perp}}t_{ij}\|_2$.

Lemma 4. Fix T, P . If $\varepsilon \leq \frac{c_1 p^3}{48}$ and $p \geq \sqrt{\frac{8}{n}}$, then $\sum_{ij \in E_g} \eta_{ij} \geq \frac{c_1 p^3}{48\varepsilon} \sum_{ij \in E_b} \eta_{ij}$.

Proof. Let $i \in V_\ell, j \in V_s$. Note that Condition 1 implies $|\{(k, \ell) \mid k \neq i; \ell \neq j; i\ell, k\ell, kj \in E(G)\}| \geq (\frac{1}{2}n_s p - 1)(\frac{1}{2}n_\ell p^2 - 1) \geq \frac{1}{8}n_\ell n_s p^3$ if $p \geq \sqrt{\frac{8}{n}}$. By Condition 6, the number of pairs $(k, \ell) \in V_\ell \times V_s$ such that at least one of the edges $i\ell, k\ell, kj$ are in E_b can be counted by considering the case when $i\ell \in E_b$ (at most $(\varepsilon n_s)n_\ell$ pairs), $kj \in E_b$ (at most $(\varepsilon n_\ell)n_s$ pairs), and $k\ell \in E_b$ (at most $\varepsilon n_s n_\ell$ pairs). Hence in total, there are at most $3\varepsilon n_s n_\ell$ such pairs. By Lemma 3, the c_1 -well-distributedness of $(T^{(0)}, P^{(0)})$ along G , and the assumption that $\varepsilon \leq \frac{c_1 p^3}{48}$, we have

$$\sum_{\substack{k \in V_\ell, \ell \in V_s \\ k \neq i, \ell \neq j \\ i\ell, k\ell, kj \in E_g}} (\eta_{i\ell} + \eta_{k\ell} + \eta_{kj}) \geq \left(c_1 \cdot \frac{1}{8} n_\ell n_s p^3 - 3\varepsilon n_\ell n_s \right) \cdot \eta_{ij} \geq \frac{c_1}{16} n_\ell n_s p^3 \cdot \eta_{ij}.$$

Therefore, if we sum the inequality above for all bad edges $ij \in E_b$, then

$$\sum_{ij \in E_b} \sum_{\substack{k \in V_\ell, \ell \in V_s \\ k \neq i, \ell \neq j \\ i\ell, k\ell, kj \in E_g}} (\eta_{i\ell} + \eta_{k\ell} + \eta_{kj}) \geq \frac{c_1}{16} n_\ell n_s p^3 \cdot \sum_{ij \in E_b} \eta_{ij}.$$

For fixed $k\ell \in E_g$, the left-hand-side may sum $\eta_{k\ell}$ as many times as the number of C_4 s in $E(G)$ that contain $k\ell$ and exactly one bad edge. This is the same as the number of C_4 s whose edge opposite $k\ell$ is bad, plus the number of C_4 s whose edge adjacent to ℓ is bad, plus the number of

C_4 s whose edge adjacent to k is bad. In each case, there are at most $\varepsilon n_\ell n_s$ such C_4 s. Hence, the left-hand-side of above is at most

$$\sum_{ij \in E_b} \sum_{\substack{k \in V_\ell, \ell \in V_s \\ k \neq i, \ell \neq j \\ i\ell, k\ell, kj \in E_g}} (\eta_{i\ell} + \eta_{k\ell} + \eta_{kj}) \leq 3\varepsilon n_\ell n_s \cdot \sum_{ij \in E_g} \eta_{ij}.$$

Therefore by combining the two inequalities above, we obtain

$$\sum_{ij \in E_b} \eta_{ij} \leq \frac{48\varepsilon}{c_1 p^3} \sum_{ij \in E_g} \eta_{ij}. \quad \square$$

The following lemma relates the rotational motions over the good graph E_g to rotational motions over the complete bipartite graph K_{n_ℓ, n_s} .

Lemma 5. *Fix T, P . If $\varepsilon \leq \frac{c_1 p^3}{48}$ and $p \geq \sqrt{\frac{8}{n}}$, then $\sum_{ij \in E_g} \eta_{ij} \geq \frac{c_1 p}{192} \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij}$.*

Proof. Let $i \in V_\ell, j \in V_s$. Note that Condition 1 implies $|\{(k, \ell) \mid k \neq i; \ell \neq j; i\ell, k\ell, kj \in E(G)\}| \geq (\frac{1}{2}n_s p - 1)(\frac{1}{2}n_\ell p^2 - 1) \geq \frac{1}{8}n_\ell n_s p^3$ if $p \geq \sqrt{\frac{8}{n}}$. Similarly as in Lemma 4, Condition 6 implies that the number of pairs $(k, \ell) \in V_\ell \times V_s$ such that at least one of the edges $i\ell, k\ell, kj$ are in E_b is at most $3\varepsilon n_\ell n_s$. By Lemma 3, the c_1 -well-distributedness of $(T^{(0)}, P^{(0)})$ along G , and the assumption that $\varepsilon \leq \frac{c_1 p^3}{48}$, we have

$$\sum_{\substack{k \in V_\ell, \ell \in V_s \\ k \neq i, \ell \neq j \\ i\ell, k\ell, kj \in E_g}} (\eta_{i\ell} + \eta_{k\ell} + \eta_{kj}) \geq \left(c_1 \cdot \frac{1}{8}n_\ell n_s p^3 - 3\varepsilon n_\ell n_s \right) \cdot \eta_{ij} \geq \frac{c_1}{16} n_\ell n_s p^3 \cdot \eta_{ij}.$$

Therefore, if we sum the inequality above for all $i \in V_\ell, j \in V_s$, or equivalently over all $ij \in E(K_{n_\ell, n_s})$, then

$$\sum_{ij \in E(K_{n_\ell, n_s})} \sum_{\substack{k \in V_\ell, \ell \in V_s \\ k \neq i, \ell \neq j \\ i\ell, k\ell, kj \in E_g}} (\eta_{i\ell} + \eta_{k\ell} + \eta_{kj}) \geq \frac{c_1}{16} n_\ell n_s p^3 \cdot \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij}.$$

For fixed $k\ell \in E_g$, the left-hand-side may sum $\eta_{k\ell}$ as many times as the number of paths of length 3 in G that contain $k\ell$. Each path of length 3 can be thought of as an edge originating from V_ℓ , an edge in the middle, and an edge terminating in V_s . The total number of paths of length 3 in G containing $k\ell$ equals the number which have $k\ell$ as the middle edge, plus the number with $k\ell$ as the edge originating from V_ℓ , plus the number with $k\ell$ as the edge terminating in V_s . In each of these cases, Condition 1 ensures that there are at most $4p^2 n_\ell n_s$ such paths of length 3. Hence, the term $\eta_{k\ell}$ appears at most $12p^2 n_\ell n_s$ times. Hence, the left-hand-side of above is at most

$$\sum_{ij \in E(K_{n_\ell, n_s})} \sum_{\substack{k \in V_\ell, \ell \in V_s \\ k \neq i, \ell \neq j \\ i\ell, k\ell, kj \in E_g}} (\eta_{i\ell} + \eta_{k\ell} + \eta_{kj}) \leq 12p^2 n_\ell n_s \cdot \sum_{ij \in E_g} \eta_{ij}.$$

Therefore by combining the two inequalities above, we obtain

$$\sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij} \leq \frac{12 \cdot 16}{c_1 p} \sum_{ij \in E_g} \eta_{ij}. \quad \square$$

2.4 Proof of Theorem 2

We now prove the deterministic recovery theorem.

Proof of Theorem 2. By Lemma 1 and the fact that Conditions 1–6 are invariant under global translation and nonzero scaling, we can take $\bar{\zeta}^{(0)} = 0$ and $L(T^{(0)}, P^{(0)}) = 1$ without loss of generality. The variable $\mu_\infty = \max_{i \neq j} \|t_{ij}^{(0)}\|_2$ is to be understood accordingly.

We will directly prove that $R(T, P) > R(T^{(0)}, P^{(0)})$ for all $(T, P) \neq (T^{(0)}, P^{(0)})$ such that $L(T, P) = 1$ and $\bar{t} + \bar{p} = 0$. Consider an arbitrary feasible T, P and recall the notation $t_{ij} = (1 + \delta_{ij})t_{ij}^{(0)} + \eta_{ij}s_{ij}$ where s_{ij} is a unit vector orthogonal to $t_{ij}^{(0)}$ and $\eta_{ij} = \|P_{t_{ij}^{(0)\perp}} t_{ij}\|_2$. Since $v_{ij} = \hat{t}_{ij}^{(0)}$ holds for all $ij \in E_g$, a useful lower bound for the objective $R(T, P)$ is given by

$$\begin{aligned} R(T, P) &= \sum_{ij \in E(G)} \|P_{v_{ij}^\perp} t_{ij}\|_2 = \sum_{ij \in E_g} \eta_{ij} + \sum_{ij \in E_b} \|P_{v_{ij}^\perp} t_{ij}\|_2 \\ &\geq \sum_{ij \in E_g} \eta_{ij} + \sum_{ij \in E_b} \left(\|P_{v_{ij}^\perp} t_{ij}^{(0)}\|_2 - |\delta_{ij}| \|t_{ij}^{(0)}\|_2 - \eta_{ij} \right) \\ &\geq R(T^{(0)}, P^{(0)}) + \sum_{ij \in E_g} \eta_{ij} - \sum_{ij \in E_b} (|\delta_{ij}| \|t_{ij}^{(0)}\|_2 + \eta_{ij}). \end{aligned} \quad (5)$$

Suppose that $\sum_{ij \in E_b} |\delta_{ij}| \|t_{ij}^{(0)}\|_2 < \sum_{ij \in E_b} \eta_{ij}$. Since Lemma 4 for $\varepsilon \leq \frac{c_1 p^3}{96}$ implies $\sum_{ij \in E_b} \eta_{ij} \leq \frac{1}{2} \sum_{ij \in E_g} \eta_{ij}$, by (5), we have

$$\begin{aligned} R(T, P) &\geq R(T^{(0)}, P^{(0)}) + \sum_{ij \in E_g} \eta_{ij} - \sum_{ij \in E_b} (|\delta_{ij}| \|t_{ij}^{(0)}\|_2 + \eta_{ij}) \\ &> R(T^{(0)}, P^{(0)}) + \sum_{ij \in E_g} \eta_{ij} - \sum_{ij \in E_b} 2\eta_{ij} \geq R(T^{(0)}, P^{(0)}). \end{aligned}$$

Hence we may assume

$$\sum_{ij \in E_b} |\delta_{ij}| \|t_{ij}^{(0)}\|_2 \geq \sum_{ij \in E_b} \eta_{ij}. \quad (6)$$

In the case $|E_b| \neq 0$, define $\bar{\delta} = \frac{1}{|E_b|} \sum_{ij \in E_b} |\delta_{ij}|$ as the average ‘relative parallel motion’ on the bad edges. For a pair of vertex-disjoint edges $ij, k\ell \in E(K_{n_\ell, n_s})$, define $\eta(ij, k\ell) = \eta_{ij} + \eta_{kj} + \eta_{k\ell} + \eta_{\ell\ell}$,

Case 0. $|E_b| = 0$ or $\bar{\delta} = 0$.

Note that $\bar{\delta} = 0$ implies $\delta_{ij} = 0$ for all $ij \in E_b$, which by (6) implies $\eta_{ij} = 0$ for all $ij \in E_b$. Therefore by (5), we have

$$R(T, P) \geq R(T^{(0)}, P^{(0)}) + \sum_{ij \in E_g} \eta_{ij}.$$

If $\sum_{ij \in E_g} \eta_{ij} > 0$, then we have $R(T, P) > R(T^{(0)}, P^{(0)})$. Thus we may assume that $\eta_{ij} = 0$ for all $ij \in E_g$. In this case, we will show that $T = T^{(0)}$ and $P = P^{(0)}$.

By Lemma 5, if $\varepsilon \leq \frac{c_1 p^3}{48}$, then $\eta_{ij} = 0$ for all $ij \in E(G)$ implies that $\eta_{ij} = 0$ for all $ij \in E(K_{n_\ell, n_s})$. For $ij \in E_b$, since $\delta_{ij} = \eta_{ij} = 0$, it follows that $\ell_{ij} = \ell_{ij}^{(0)}$. Since $\delta_{ij} \|t_{ij}^{(0)}\|_2 = \ell_{ij} - \ell_{ij}^{(0)}$ for $ij \in E_g$, we have

$$0 = \sum_{ij \in E(G)} (\ell_{ij} - \ell_{ij}^{(0)}) = \sum_{ij \in E_b} (\ell_{ij} - \ell_{ij}^{(0)}) + \sum_{ij \in E_g} (\ell_{ij} - \ell_{ij}^{(0)}) = \sum_{ij \in E_g} (\ell_{ij} - \ell_{ij}^{(0)}) = \sum_{ij \in E_g} \delta_{ij} \|t_{ij}^{(0)}\|_2,$$

where the first equality is because $L(T, P) = L(T^{(0)}, P^{(0)}) = 1$. By Condition 2, $\|t_{ij}^{(0)}\|_2 \neq 0$ for all $i \neq j$. Therefore, if $\delta_{ij} \neq 0$ for some $ij \in E_g$, then there exists $ab, cd \in E_g$ such that $\delta_{ab} > 0$ and $\delta_{cd} < 0$. If ab and cd are vertex-disjoint, Lemma 2 and Conditions 2 and 4 force $\eta(ab, cd) > 0$, which contradicts the fact that $\eta_{ij} = 0$ for all $ij \in E(K_{n_\ell, n_s})$. If ab and cd are not vertex-disjoint, then, let $abc'd'$ be an arbitrary C_4 containing ab and cd . Then Lemma 2 implies the same result as above. Therefore $\delta_{ij} = 0$ for all $ij \in E_g$, and hence $\delta_{ij} = 0$ for all $ij \in E(G)$.

Define $t_i = t_i^{(0)} + h_i$ for each $i \in V_\ell$. Define $p_j = p_j^{(0)} + h_j$ for $j \in V_s$. Because $\eta_{ij} = \delta_{ij} = 0$ for all $ij \in E(G)$, we have $h_i = h_j$ for all $ij \in E(G)$. Since G is connected by Condition 1, this implies $h_i = h_j$ for all $i \in V_\ell, j \in V_s$. Then by the constraint $\sum_{i \in V_\ell} t_i + \sum_{j \in V_s} p_j = 0$, we get $h_i = 0$ for all $i \in V_\ell$ and $h_j = 0$ for all $j \in V_s$. Therefore $T = T^{(0)}$ and $P = P^{(0)}$.

Case 1. $|E_b| \neq 0$ and $\bar{\delta} \neq 0$ and $\sum_{ij \in E_g} |\delta_{ij}| < \frac{1}{8}\bar{\delta}|E_g|$.

Define $L_b = \{ij \in E_b : |\delta_{ij}| \geq \frac{1}{2}\bar{\delta}\}$. Note that $\sum_{ij \in E_b \setminus L_b} |\delta_{ij}| < \frac{1}{2}\bar{\delta}|E_b|$ and therefore

$$\sum_{ij \in L_b} |\delta_{ij}| = \sum_{ij \in E_b} |\delta_{ij}| - \sum_{ij \in E_b \setminus L_b} |\delta_{ij}| > \sum_{ij \in E_b} |\delta_{ij}| - \frac{1}{2}\bar{\delta}|E_b| = \frac{1}{2}\bar{\delta}|E_b|. \quad (7)$$

Define $F_g = \{ij \in E_g : |\delta_{ij}| < \frac{1}{4}\bar{\delta}\}$. Then by the condition of Case 1,

$$\frac{1}{8}\bar{\delta}|E_g| > \sum_{ij \in E_g} |\delta_{ij}| \geq \sum_{ij \in E_g \setminus F_g} |\delta_{ij}| \geq \frac{1}{4}\bar{\delta}|E_g \setminus F_g|,$$

and therefore $|E_g \setminus F_g| < \frac{1}{2}|E_g|$, or equivalently, $|F_g| > \frac{1}{2}|E_g|$.

For each $ij \in L_b$, define $F_g(i, j) = \{k\ell \in F_g \mid k \neq i, \ell \neq j\}$. Note that by Condition 1, $|F_g(i, j)| > \frac{1}{2}|E_g| - 2p(n_\ell + n_s)$. For any $k\ell \in F_g(i, j)$, since $|\delta_{ij}| \geq \frac{1}{2}\bar{\delta}$ and $|\delta_{k\ell}| < \frac{1}{4}\bar{\delta}$, we have $|\delta_{ij} - \delta_{k\ell}| \geq \frac{1}{2}|\delta_{ij}|$. Thus Lemma 2 and Conditions 3 and 4 give $\eta(ij, k\ell) \geq \beta|\delta_{ij} - \delta_{k\ell}| \cdot \|t_{ij}^{(0)}\|_2 \geq \beta \cdot \frac{1}{2}|\delta_{ij}| \cdot \|t_{ij}^{(0)}\|_2 \geq \frac{\beta c_0 \mu_\infty}{2}|\delta_{ij}|$. Therefore by Condition 1,

$$\begin{aligned} \sum_{ij \in E_b} \sum_{\substack{k\ell \in E_g \\ k \neq i, \ell \neq j}} \eta(ij, k\ell) &\geq \sum_{ij \in L_b} \sum_{k\ell \in F_g(i, j)} \frac{\beta c_0 \mu_\infty}{2} |\delta_{ij}| = \sum_{ij \in L_b} |F_g(i, j)| \cdot \frac{\beta c_0 \mu_\infty}{2} |\delta_{ij}| \\ &> \sum_{ij \in L_b} \frac{\beta c_0 \mu_\infty}{2} \left(\frac{1}{2}|E_g| - 2p(n_\ell + n_s) \right) |\delta_{ij}| \end{aligned}$$

Note that if $\varepsilon < \frac{1}{4}p$, then $|E_g| \geq \frac{n_\ell n_s p}{2} - |E_b| \geq \frac{n_\ell n_s p}{4}$. Further note that $n_\ell, n_s > 64$ implies that $2p(n_\ell + n_s) < \frac{1}{16}n_\ell n_s p$. Hence by (7),

$$\sum_{ij \in E_b} \sum_{\substack{k\ell \in E_g \\ k \neq i, \ell \neq j}} \eta(ij, k\ell) \geq \frac{\beta c_0 \mu_\infty}{32} n_\ell n_s \cdot \sum_{ij \in L_b} |\bar{\delta}_{ij}| \geq \frac{\beta c_0 \mu_\infty}{32} n_\ell n_s \cdot \frac{1}{2}\bar{\delta}|E_b|.$$

For each $ij \in E(K_{n_\ell, n_s})$, we would like to count how many times each η_{ij} appear on the left hand side. If $ij \in E_b$, then there are at most $n_\ell n_s$ C_4 s containing ij ; hence η_{ij} may appear at most $4n_\ell n_s$ times. If $ij \notin E_b$, then η_{ij} appears when there is a C_4 containing ij and some bad edge. If the bad edge is incident to ij , then there are at most $2\varepsilon n_\ell n_s$ such C_4 s, and if the bad edge is not incident to

ij , then there are at most $|E_b| \leq \varepsilon n_\ell n_s$ such C_4 s. Thus η_{ij} may appear at most $4 \cdot 3\varepsilon n_\ell n_s = 12\varepsilon n_\ell n_s$ times. Therefore

$$\sum_{ij \in E_b} \sum_{k\ell \in E_g} \eta(ij, k\ell) \leq \sum_{ij \in E_b} 4n_\ell n_s \cdot \eta_{ij} + \sum_{ij \in E(K_{n_\ell, n_s})} 12\varepsilon n_\ell n_s \cdot \eta_{ij}.$$

By Lemma 4, if $\varepsilon < \frac{c_1 p^3}{48}$, we have

$$\sum_{ij \in E_b} \sum_{k\ell \in E_g} \eta(ij, k\ell) \leq \frac{48 \cdot 4\varepsilon}{c_1 p^3} n_\ell n_s \sum_{ij \in E_g} \eta_{ij} + \sum_{ij \in E(K_{n_\ell, n_s})} 12\varepsilon n_\ell n_s \cdot \eta_{ij} \leq \frac{204\varepsilon}{c_1 p^3} n_\ell n_s \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij}.$$

Hence

$$\frac{204\varepsilon}{c_1 p^3} n_\ell n_s \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij} \geq \frac{\beta c_0 \mu_\infty}{64} n_\ell n_s \cdot \bar{\delta} |E_b|.$$

If $\varepsilon < \frac{\beta c_0 c_1^2 p^4}{384 \cdot 204 \cdot 64}$, then by Condition 3, $\bar{\delta} \neq 0$, and $|E_b| \neq 0$, the above implies

$$\begin{aligned} \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij} &\geq \frac{\beta c_0 c_1 p^3}{204 \cdot 64 \varepsilon} \mu_\infty \cdot \bar{\delta} |E_b| \\ &> \frac{384}{c_1 p} \mu_\infty \cdot \bar{\delta} |E_b| \geq \frac{384}{c_1 p} \sum_{ij \in E_b} |\delta_{ij}| \|t_{ij}^{(0)}\|_2. \end{aligned}$$

Lemma 5 implies

$$\sum_{ij \in E_g} \eta_{ij} \geq \frac{c_1 p}{192} \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij} > 2 \sum_{ij \in E_b} |\delta_{ij}| \|t_{ij}^{(0)}\|_2.$$

Therefore by (6), we have $\sum_{ij \in E_g} \eta_{ij} > \sum_{ij \in E_b} (|\delta_{ij}| \|t_{ij}^{(0)}\|_2 + \eta_{ij})$ if $\varepsilon \leq \min\{\frac{c_1 p^3}{48}, \frac{p}{4}, \frac{\beta c_0 c_1^2 p^4}{384 \cdot 204 \cdot 64}\}$ and $p \geq \sqrt{\frac{8}{n}}$. By (5), this shows $R(T, P) > R(T^{(0)}, P^{(0)})$. This condition on ε is satisfied under the assumption $\varepsilon \leq \frac{\beta c_0 c_1^2 p^4}{384 \cdot 204 \cdot 64}$.

Case 2. $|E_b| \neq 0$ and $\bar{\delta} \neq 0$ and $\sum_{ij \in E_g} |\delta_{ij}| \geq \frac{1}{8} \bar{\delta} |E_g|$.

Define $E_+ = \{ij \in E_g : \delta_{ij} \geq 0\}$ and $E_- = \{ij \in E_g : \delta_{ij} < 0\}$. Since $\ell_{ij} - \ell_{ij}^{(0)} = \delta_{ij} \|t_{ij}^{(0)}\|_2$ for $ij \in E_g$, we have

$$0 = \sum_{ij \in E(G)} (\ell_{ij} - \ell_{ij}^{(0)}) = \sum_{ij \in E_b} (\ell_{ij} - \ell_{ij}^{(0)}) + \sum_{ij \in E_g} \delta_{ij} \|t_{ij}^{(0)}\|_2.$$

where the first equality follows from $L(T, P) = L(T^{(0)}, P^{(0)})$. Therefore,

$$\left| \sum_{ij \in E_g} \delta_{ij} \|t_{ij}^{(0)}\|_2 \right| \leq \left| \sum_{ij \in E_b} (\ell_{ij} - \ell_{ij}^{(0)}) \right| \leq \sum_{ij \in E_b} (|\delta_{ij}| \|t_{ij}^{(0)}\|_2 + \eta_{ij}) \leq 2\mu_\infty \bar{\delta} |E_b|,$$

where the last inequality follows from (6), Condition 3, and the definition of $\bar{\delta}$. On the other hand, the condition of Case 2 and Condition 3 implies $\sum_{ij \in E_g} |\delta_{ij}| \|t_{ij}^{(0)}\|_2 \geq \frac{1}{8} c_0 \mu_\infty \bar{\delta} |E_g|$. Therefore

$$\sum_{ij \in E_-} (-\delta_{ij}) \|t_{ij}^{(0)}\|_2 = \frac{1}{2} \left(- \sum_{ij \in E_g} \delta_{ij} \|t_{ij}^{(0)}\|_2 + \sum_{ij \in E_g} |\delta_{ij}| \|t_{ij}^{(0)}\|_2 \right) \geq \frac{1}{2} \left(\frac{1}{8} c_0 \mu_\infty \bar{\delta} |E_g| - 2\mu_\infty \bar{\delta} |E_b| \right).$$

If $\varepsilon \leq \frac{1}{128}c_0p$, then since $|E_b| \leq \varepsilon n_\ell n_s$ and $|E_g| \geq \frac{n_\ell n_s p}{2} - |E_b| \geq \frac{n_\ell n_s p}{4}$, we see that $\frac{1}{8}c_0\mu_\infty\bar{\delta}|E_g| - 2\mu_\infty\bar{\delta}|E_b| \geq \frac{1}{16}c_0\mu_\infty\bar{\delta}|E_g|$. Therefore $\sum_{ij \in E_-} (-\delta_{ij})\|t_{ij}^{(0)}\|_2 \geq \frac{1}{32}c_0\mu_\infty\bar{\delta}|E_g|$. Similarly, $\sum_{ij \in E_+} \delta_{ij}\|t_{ij}^{(0)}\|_2 \geq \frac{1}{32}c_0\mu_\infty\bar{\delta}|E_g|$.

If $|E_+| \geq \frac{1}{2}|E_g|$, then by Lemma 2 and Conditions 3 and 4, we have

$$\begin{aligned} \sum_{ij \in E_-} \sum_{\substack{k\ell \in E_+ \\ k \neq i, \ell \neq j}} \eta(ij, k\ell) &\geq \sum_{ij \in E_-} \sum_{\substack{k\ell \in E_+ \\ k \neq i, \ell \neq j}} \beta(-\delta_{ij})\|t_{ij}^{(0)}\|_2 \\ &\geq \sum_{ij \in E_-} (-\delta_{ij})\|t_{ij}^{(0)}\|_2 \cdot \beta(|E_+| - 2p(n_\ell + n_s)) \\ &\geq \frac{1}{32}c_0\mu_\infty\bar{\delta}|E_g| \cdot \beta(|E_+| - 2p(n_\ell + n_s)) \\ &\geq \frac{\beta}{32}c_0\mu_\infty\bar{\delta}|E_g| \left(\frac{1}{2}|E_g| - 2p(n_\ell + n_s) \right). \end{aligned}$$

Note that if $\varepsilon < \frac{1}{4}p$, then $|E_g| \geq \frac{n_\ell n_s p}{2} - |E_b| \geq \frac{n_\ell n_s p}{4}$. Further note that $n_\ell, n_s > 64$ implies that $2p(n_\ell + n_s) < \frac{1}{16}n_\ell n_s p$. Hence,

$$\sum_{ij \in E_-} \sum_{\substack{k\ell \in E_+ \\ k \neq i, \ell \neq j}} \eta(ij, k\ell) \geq \frac{1}{32}\beta c_0\mu_\infty\bar{\delta} \cdot \frac{n_\ell n_s p}{4} \cdot \frac{n_\ell n_s p}{16} \geq \frac{\beta c_0\mu_\infty\bar{\delta}n_\ell^2 n_s^2 p^2}{32 \cdot 64}.$$

Similarly, if $|E_-| \geq \frac{1}{2}|E_g|$, then we can switch the order of summation and consider $\sum_{ij \in E_+} \sum_{k\ell \in E_-} \eta(ij, k\ell)$ to obtain the same conclusion.

Since each edge is contained in at most $n_\ell n_s$ copies of C_4 and there are 4 edges in a C_4 , we have

$$\sum_{ij \in E_-} \sum_{\substack{k\ell \in E_+ \\ k \neq i, \ell \neq j}} \eta(ij, k\ell) \leq 4n_\ell n_s \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij}.$$

If $\varepsilon < \frac{\beta c_0 c_1 p^3}{384 \cdot 4 \cdot 32 \cdot 64}$, then since $\bar{\delta} \neq 0$ and $|E_b| \leq \varepsilon n_\ell n_s$, we have

$$\sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij} \geq \frac{1}{4n_\ell n_s} \cdot \frac{\beta c_0\mu_\infty\bar{\delta}}{32 \cdot 64} n_\ell^2 n_s^2 p^2 \geq \frac{\beta c_0 p^2}{4 \cdot 32 \cdot 64} \mu_\infty \bar{\delta} n_\ell n_s > \frac{384}{c_1 p} \mu_\infty \bar{\delta} |E_b|.$$

By Lemma 5, if $\varepsilon < \frac{c_1 p^3}{48}$, then this implies

$$\sum_{ij \in E_g} \eta_{ij} \geq \frac{c_1 p}{192} \sum_{ij \in E(K_{n_\ell, n_s})} \eta_{ij} > 2\mu_\infty \bar{\delta} |E_b|.$$

Therefore from (5), (6), and Condition 3, if $\varepsilon \leq \min\{\frac{c_0 p}{128}, \frac{c_1 p^3}{96}, \frac{p}{4}, \frac{\beta c_0 c_1 p^3}{384 \cdot 4 \cdot 32 \cdot 64}\}$ and $p \geq \sqrt{\frac{8}{n}}$, then

$$\begin{aligned} R(T, P) &\geq R(T^{(0)}, P^{(0)}) + \sum_{ij \in E_g} \eta_{ij} - \sum_{ij \in E_b} (|\delta_{ij}|\|t_{ij}^{(0)}\|_2 + \eta_{ij}) \\ &> R(T^{(0)}, P^{(0)}) + 2\mu_\infty \bar{\delta} |E_b| - \sum_{ij \in E_b} 2|\delta_{ij}|\|t_{ij}^{(0)}\|_2 \geq R(T^{(0)}, P^{(0)}). \end{aligned}$$

This condition on ε is satisfied under the assumption $\varepsilon \leq \frac{\beta c_0 c_1^2 p^4}{384 \cdot 204 \cdot 64}$. \square

2.5 Properties of Gaussians

In this section, we prove that i.i.d. Gaussians satisfy properties needed to establish Conditions 3–5 in Theorem 2. We begin by recording some useful facts regarding concentration of random Gaussian vectors:

Lemma 6. *Let x, y be i.i.d. $\mathcal{N}(0, I_{d \times d})$, and $\epsilon \leq 1$, then*

$$\mathbb{P}(d(1 - \epsilon) \leq \|x\|_2^2 \leq d(1 + \epsilon)) \geq 1 - e^{-c\epsilon^2 d}$$

and

$$\mathbb{P}(|\langle x, y \rangle| \geq d\epsilon) \leq e^{-c\epsilon^2 d}$$

where $c > 0$ is an absolute constant.

Proof. Both statements follow from Corollary 5.17 in [10], concerning concentration of sub-exponential random variables. \square

Lemma 7. *Corollary 5.35 in [10]. Let A be an $n \times d$ matrix with iid $\mathcal{N}(0, 1)$ entries. Then for any $t \geq 0$,*

$$\mathbb{P}(\sigma_{\max}(A) \geq \sqrt{n} + \sqrt{d} + t) \leq 2e^{-\frac{t^2}{2}}$$

where $\sigma_{\max}(A)$ is the largest singular value of A .

Lemma 8. *Let $t_i, t_j, t_k, t_\ell \sim \mathcal{N}(0, I_{d \times d})$ be independent. There is a universal constant c such that with probability at least $1 - 15e^{-cd}$,*

$$\frac{\|P_{\text{span}(t_k - t_j, t_i - t_\ell)^\perp}(t_i - t_j)\|}{\|t_i - t_j\|} \geq \frac{1}{4}.$$

Proof. Let c be the constant from Lemma 6. Let $x = t_i - t_\ell, y = t_k - t_j, z = t_i - t_j$. Observe

$$\begin{aligned} P_{\text{span}(x, y)^\perp} \hat{z} &= \hat{z} - \langle \hat{z}, \hat{x} \rangle \hat{x} - \langle \hat{z}, \hat{y}_{x^\perp} \rangle \hat{y}_{x^\perp} = \hat{z} - \langle \hat{z}, \hat{x} \rangle \hat{x} - \langle \hat{z}, \hat{y} \rangle \hat{y} + (\langle \hat{z}, \hat{y} \rangle \hat{y} - \langle \hat{z}, \hat{y}_{x^\perp} \rangle \hat{y}_{x^\perp}) \\ &= \hat{z} - \langle \hat{z}, \hat{x} \rangle \hat{x} - \langle \hat{z}, \hat{y} \rangle \hat{y} + (\hat{y} \hat{y}^t - \hat{y}_{x^\perp} \hat{y}_{x^\perp}^t) \hat{z}, \end{aligned}$$

where $\hat{y}_{x^\perp} = \frac{y - \langle y, \hat{x} \rangle \hat{x}}{\|y - \langle y, \hat{x} \rangle \hat{x}\|}$, which is well defined with probability 1. By the triangle inequality,

$$\|P_{\text{span}(x, y)^\perp} \hat{z}\| \geq \sqrt{1 - |\langle \hat{z}, \hat{x} \rangle|^2} - |\langle \hat{z}, \hat{y} \rangle| - \|\hat{y} \hat{y}^t - \hat{y}_{x^\perp} \hat{y}_{x^\perp}^t\|_{\text{op}}$$

For arbitrary unit vectors $\hat{a}, \hat{b} \in \mathbb{R}^d$, $\|\hat{a} \hat{a}^t - \hat{b} \hat{b}^t\|_{\text{op}} = |\sin \theta|$, where θ is the angle between \hat{a} and \hat{b} . This fact can be verified by direct computation after taking $\hat{a} = e_1$ and $\hat{b} = \cos \theta e_1 + \sin \theta e_2$ without loss of generality. Hence, $\|\hat{y} \hat{y}^t - \hat{y}_{x^\perp} \hat{y}_{x^\perp}^t\|_{\text{op}} = |\sin \theta| = |\cos \alpha|$, where θ is the angle between \hat{y} and \hat{y}_{x^\perp} , and α is the angle between \hat{y} and \hat{x} . Thus $\|\hat{y} \hat{y}^t - \hat{y}_{x^\perp} \hat{y}_{x^\perp}^t\|_{\text{op}} = |\langle \hat{y}, \hat{x} \rangle|$. So,

$$\|P_{\text{span}(x, y)^\perp} \hat{z}\| \geq \sqrt{1 - |\langle \hat{z}, \hat{x} \rangle|^2} - |\langle \hat{z}, \hat{y} \rangle| - |\langle \hat{y}, \hat{x} \rangle|$$

Now, note that

$$|\langle \hat{z}, \hat{x} \rangle|^2 = \frac{\langle t_i - t_j, t_i - t_\ell \rangle^2}{\|t_i - t_j\|^2 \|t_i - t_\ell\|^2} = \frac{(\|t_i\|^2 - \langle t_i, t_\ell \rangle - \langle t_j, t_i \rangle + \langle t_j, t_\ell \rangle)^2}{\|t_i - t_j\|^2 \|t_i - t_\ell\|^2}$$

By Lemma 6 with $\varepsilon = 0.01$,

$$|\langle \hat{z}, \hat{x} \rangle|^2 \leq \frac{(d(1+\varepsilon) + 3d\varepsilon)^2}{4d^2(1-\varepsilon)^2} \leq 0.3$$

with probability at least $1 - 6e^{-cd}$ for some universal constant c . Similarly, $|\langle \hat{z}, \hat{y} \rangle|^2 \leq 0.3$ with the same probability. Since \hat{y} and \hat{x} are independent, by Lemma 6 with $\varepsilon = 0.01$, $|\langle \hat{y}, \hat{x} \rangle| \leq \frac{\varepsilon d}{d(1-\varepsilon)} \leq 2\varepsilon$ with probability at least $1 - 3e^{-cd}$. Thus, we observe

$$\|P_{\text{span}(x,y)^\perp} \hat{z}\| \geq \sqrt{1-0.3} - \sqrt{0.3} - 0.02 \geq \frac{1}{4}$$

with probability at least $1 - 15e^{-cd}$. \square

Lemma 9. *Let $t_i, t_j, t_k, t_\ell \sim \mathcal{N}(0, I_{d \times d})$ be independent for $d \geq 3$. There is a universal constant c such that with probability at least $1 - 7e^{-cd}$,*

$$\frac{\|P_{\text{span}(t_i-t_\ell, t_k-t_\ell)^\perp}(t_i-t_j)\|}{\|t_i-t_j\|_2} \geq \frac{1}{4}.$$

Proof. Let c be the constant from Lemma 6. Let $u = \frac{t_i-t_\ell}{\sqrt{2}}, v = \frac{t_k-t_\ell}{\sqrt{2}}, w = t_j, x = t_k$. Each of these variables are i.i.d. $\mathcal{N}(0, 1)$. Note that

$$P_{\text{span}(t_i-t_\ell, t_k-t_\ell)^\perp}(t_i-t_j) = -P_{\text{span}(u, x-\frac{v}{\sqrt{2}})^\perp}\left(w - \frac{v}{\sqrt{2}}\right).$$

Without loss of generality, rotate coordinates so that u is in the direction of e_1 . Thus, it suffices to bound $\|P_{(\tilde{x}-\frac{\tilde{v}}{\sqrt{2}})^\perp}(\tilde{w} - \frac{\tilde{v}}{\sqrt{2}})\|_2$ where $\tilde{v}, \tilde{w}, \tilde{x} \sim \mathcal{N}(0, I_{d-1 \times d-1})$. Note that $\tilde{x} - \frac{\tilde{v}}{\sqrt{2}}$ and $\tilde{w} - \frac{\tilde{v}}{\sqrt{2}}$ both follow the distribution $\mathcal{N}(0, \frac{3}{2}I_{d-1 \times d-1})$. Note that

$$\begin{aligned} \left\|P_{(\tilde{x}-\frac{\tilde{v}}{\sqrt{2}})^\perp}\left(\tilde{w} - \frac{\tilde{v}}{\sqrt{2}}\right)\right\|^2 &= \left\|\tilde{w} - \frac{\tilde{v}}{\sqrt{2}}\right\|^2 - \frac{\langle \tilde{w} - \frac{\tilde{v}}{\sqrt{2}}, \tilde{x} - \frac{\tilde{v}}{\sqrt{2}} \rangle^2}{\|\tilde{x} - \frac{\tilde{v}}{\sqrt{2}}\|^2} \\ &= \left\|\tilde{w} - \frac{\tilde{v}}{\sqrt{2}}\right\|^2 - \frac{\left(\langle \tilde{w}, \tilde{x} \rangle - \frac{1}{\sqrt{2}}\langle \tilde{w}, \tilde{v} \rangle - \frac{1}{\sqrt{2}}\langle \tilde{v}, \tilde{x} \rangle + \frac{\|\tilde{v}\|^2}{2}\right)^2}{\|\tilde{x} - \frac{\tilde{v}}{\sqrt{2}}\|^2}. \end{aligned}$$

Hence Lemma 6 with $\varepsilon = 0.01$ shows that with probability at least $1 - 6e^{-cd}$, the above is at least

$$\frac{3}{2}(d-1)(1-\varepsilon) - \frac{\left(\frac{1}{2}(d-1)(1+\varepsilon) - 3\varepsilon(d-1)\right)^2}{\frac{3}{2}(d-1)(1-\varepsilon)} \geq (d-1) \geq \frac{2}{3}d,$$

Thus, we have that

$$\|P_{\text{span}(t_j-t_\ell, t_k-t_\ell)^\perp}(t_i-t_j)\|^2 \geq \frac{2}{3}d$$

with probability at least $1 - 6e^{-cd}$. To conclude the proof, note that Lemma 6 with $\varepsilon = 0.01$ implies that $\|t_i - t_j\|^2 \geq 2d(1+\varepsilon)$ with probability at least $1 - e^{-cd}$. \square

We can now establish Conditions 3–4 of Theorem 2 with high probability.

Lemma 10. Let $t_i, p_j \sim \mathcal{N}(0, I_{d \times d})$ for $i \in V_\ell, j \in V_s$ be independent. Condition 3 of Theorem 2 holds with $c_0 = \frac{9}{10}$ with probability at least $1 - 2n_\ell n_s e^{-cd}$ for a universal constant c . Condition 4 of Theorem 2 holds with $\beta = \frac{1}{4}$ with probability at least $1 - 22n_\ell^2 n_s^2 e^{-cd}$.

Proof. Condition 3 follows from applying Lemma 6 with $\varepsilon = 0.01$ and a union bound to $\|t_i - p_j\|_2$ for all $n_\ell n_s$ pairs $(i, j) \in V_\ell \times V_s$. Condition 4 follows from applying Lemmas 8 and 9 and a union bound over the at most $n_\ell^2 n_s^2$ choices of $i, k \in V_\ell$ and $j, \ell \in V_s$. \square

2.6 Gaussians are well-distributed

In this section we prove that Condition 5 of Theorem 2 holds with high probability.

Lemma 11. There exist constants d_0 and K_0 such that the following holds. Let $G = (V_\ell \cup V_s, E)$ be a bipartite- p -typical graph. Let $t_i, p_j \sim \mathcal{N}(0, I_{d \times d})$ for $i \in V_\ell, j \in V_s$ be independent from G and each other. Let $T = \{t_i\}_{i \in V_\ell}$ and $P = \{p_j\}_{j \in V_s}$. If $d \geq d_0$ and $n_\ell, n_s \geq \max(K_0, 160d)$, then (T, P) is $\frac{1}{20}$ -well-distributed along G with probability at least $1 - O(n_\ell^2 n_s^2 e^{-cd})$ for universal constants c, K_0 .

We start by proving an intermediate lemma asserting the well-distributedness of pairs of random Gaussian vectors $\{(t_i, p_i)\}_{i \in [k]}$ with respect to a fixed pair of random Gaussian vectors (x, y) .

Lemma 12. There exist positive constants d_0, \tilde{K}_0 such that the following holds. Let $x, y, t_i, p_i \sim \mathcal{N}(0, I_{d \times d})$ be independent, where $i \in [k]$. Then the set $\{(t_i, p_i)\}_{i \in [k]}$ is $\frac{1}{10}$ -well-distributed with respect to (x, y) with probability $1 - 6ke^{-cd}$ if $k \geq \max(\tilde{K}_0, 10d)$ and $d \geq d_0$.

The proof of this lemma appears at the end of this section. We will deduce Lemma 11 from Lemma 12 by partitioning the edge set of G into sets of vertex-disjoint edges. A *matching* is a set of vertex-disjoint edges. A *perfect matching* of a graph is a matching that intersects all vertices. The following is a well-known lemma in Graph theory.

Lemma 13. Let $G = (V, E)$ be a bipartite graph with vertex partition $V = V_1 \cup V_2$, and let Δ be the maximum degree of G . There exists an edge-partition $E = E_1 \cup \dots \cup E_\Delta$ such that E_a forms a matching for each $a \in [\Delta]$.

Proof. By adding vertices and edges to G if necessary, we can obtain a Δ -regular bipartite multi-graph G' . By Hall's theorem, every non-empty regular multi-graph contains a perfect matching (see [2, Corollary 2.1.3]). Let F_1 be an arbitrary perfect matching of G' . Remove F_1 from the edge set of G' , and note that the remaining graph is still regular. Thus we can repeat the process to obtain a partition $E(G') = F_1 \cup \dots \cup F_\Delta$ of the edge set of G' into perfect matchings. The sets $E_a = F_a \cap E(G)$ for $a \in [\Delta]$ satisfy the claimed condition. \square

The proof of Lemma 11 follows from the two lemmas above.

Proof of Lemma 11. Recall the notation that $N = \max\{|V_\ell|, |V_s|\}$ and $n = \min\{|V_\ell|, |V_s|\}$. Since G is a bipartite- p -typical graph, the maximum degree Δ of G is at most $2Np$. By Lemma 13, there exists an edge-partition $E = E_1 \cup \dots \cup E_\Delta$ such that each E_a for $a = 1, 2, \dots, \Delta$ forms a matching.

Fix a pair of indices (i_0, j_0) for $i_0 \in V_\ell$ and $j_0 \in V_s$. Let $E' \subseteq E$ be the subset of edges that do not intersect i_0 or j_0 , and for each $a \in [\Delta]$, let $E'_a \subseteq E_a$ be the subset of edges that do not intersect i_0 or j_0 . Let $A \subseteq [\Delta]$ be the set of indices a for which $|E'_a| \geq \max(\tilde{K}_0, 10d)$. For each $a \in A$, by Lemma 12, we see that with probability at least $1 - O(|E'_a|e^{-cd})$,

$$\sum_{ij \in E'_a} \|P_{\text{span}\{p_{j_0} - t_i, t_i - p_j, p_j - t_{i_0}\}^\perp}(h)\|_2 \geq \frac{1}{10} |E'_a| \|h\|_2$$

holds for all $h \in \mathbb{R}^d$. Therefore by the union bound, with probability at least $1 - O(\sum_{a \in A} |E'_a| e^{-cd}) \geq 1 - O(|E'| e^{-cd}) \geq 1 - O(nN e^{-cd})$, the above holds simultaneously for all $a \in [\Delta]$. Conditioned on this event, for all $h \in \mathbb{R}^d$,

$$\begin{aligned} & \sum_{ij \in E'} \|P_{\text{span}\{p_{j_0} - t_i, t_i - p_j, p_j - t_{i_0}\}^\perp}(h)\|_2 \\ & \geq \sum_{a \in A} \sum_{ij \in E'_a} \|P_{\text{span}\{p_{j_0} - t_i, t_i - p_j, p_j - t_{i_0}\}^\perp}(h)\|_2 \geq \sum_{a \in A} \frac{1}{10} |E'_a| \|h\|_2. \end{aligned}$$

Since $|E'| = \sum_{a=1}^\Delta |E'_a|$, we see that

$$\sum_{ij \in E'} \|P_{\text{span}\{p_{j_0} - t_i, t_i - p_j, p_j - t_{i_0}\}^\perp}(h)\|_2 \geq \frac{1}{10} \left(|E'| - \sum_{a \notin A} |E'_a| \right) \|h\|_2. \quad (8)$$

Since G is bipartite- p -typical, we have $|E'| \geq \frac{1}{2} nNp - 2(N+n)p \geq \frac{1}{4} nNp$ if $n > 16$, and by the definition of A , we have $\sum_{a \notin A} |E'_a| \leq \max(\tilde{K}_0, 10d) \cdot \Delta \leq \frac{1}{8} nNp$ if $n \geq 16 \cdot \max(\tilde{K}_0, 10d)$. Hence the right-hand-side of (8) is at least $\frac{1}{20} |E'| \|h\|_2$ for all $h \in \mathbb{R}^d$. This shows that the set $\{(t_i, p_j)\}_{i \neq i_0, j \neq j_0}$ is $\frac{1}{20}$ -well-distributed with respect to (t_{i_0}, p_{j_0}) with probability at least $1 - O(nN e^{-cd})$. By taking the union bound over all choices of pairs $(i_0, j_0) \in V_\ell \times V_s$, we can conclude that (T, P) is $\frac{1}{20}$ -well-distributed along G with probability at least $1 - O(n^2 N^2 e^{-cd})$. \square

We now prove Lemma 12.

Proof of Lemma 12. Throughout the proof, the positive constant c may change from line to line, but is always bounded below by the positive constant of the lemma statement.

For each i , let $W_i = \text{span}(t_i - y, p_i - x, t_i - p_i) = \text{span}(x - y, p_i + t_i - (x + y), t_i - p_i)$. Thus $P_{W_i^\perp} \circ P_{(x-y)^\perp} = P_{W_i^\perp}$. Therefore, it is enough to show that for all $h \perp x - y$, with high probability

$$\sum_{i=1}^n \|P_{W_i^\perp}(h)\|_2 \geq \frac{1}{10} n \|h\|_2.$$

Letting $V_i = \text{span}(x - y, p_i + t_i - x - y)$, we have

$$W_i = \text{span}(x - y, p_i + t_i - x - y, t_i - p_i) = \text{span}(x - y, p_i + t_i - x - y, P_{V_i^\perp}(t_i - p_i)).$$

Now, for any $h \perp (x - y)$,

$$\begin{aligned} \sum_{i=1}^n \|P_{W_i^\perp}(h)\|_2 & \geq \left\| \sum_{i=1}^n P_{W_i^\perp}(h) \right\|_2 \\ & = \left\| \sum_{i=1}^n \left(P_{V_i^\perp}(h) - P_{P_{V_i^\perp}(t_i - p_i)}(h) \right) \right\|_2 \\ & \geq \left\| \sum_{i=1}^n P_{V_i^\perp}(h) \right\|_2 - \left\| \sum_{i=1}^n P_{P_{V_i^\perp}(t_i - p_i)}(h) \right\|_2 \\ & \geq \left\| \sum_{i=1}^n P_{V_i^\perp}(h) \right\|_2 - \sum_{i=1}^n \left\| P_{P_{V_i^\perp}(t_i - p_i)}(h) - P_{(t_i - p_i)}(h) \right\|_2 - \left\| \sum_{i=1}^n P_{(t_i - p_i)}(h) \right\|_2. \end{aligned}$$

Since $\|P_v(h) - P_w(h)\|_2 \leq \|\hat{v}\hat{v}^t - \hat{w}\hat{w}^t\|_{\text{op}}\|h\|_2 \leq \|\hat{v} - \hat{w}\|_2\|h\|_2$ holds for all vectors $v, w, h \in \mathbb{R}^d$, the above is at least

$$\left\| \sum_{i=1}^n P_{V_i^\perp}(h) \right\|_2 - \|h\|_2 \sum_{i=1}^n \left\| \frac{P_{V_i^\perp}(t_i - p_i)}{\|P_{V_i^\perp}(t_i - p_i)\|_2} - \frac{(t_i - p_i)}{\|t_i - p_i\|_2} \right\|_2 - \left\| \sum_{i=1}^n P_{(t_i - p_i)}(h) \right\|_2. \quad (9)$$

Note that when $v = P(w)$ for some orthogonal projection operator P , we have

$$\|\hat{v} - \hat{w}\|_2^2 = 2(1 - \langle \hat{v}, \hat{w} \rangle) = 2 \left(1 - \frac{\langle P(w), w \rangle}{\|P(w)\|_2 \|w\|_2} \right) = 2 \left(1 - \frac{\|P(w)\|_2}{\|w\|_2} \right).$$

Thus,

$$\left\| \frac{P_{V_i^\perp}(t_i - p_i)}{\|P_{V_i^\perp}(t_i - p_i)\|_2} - \frac{(t_i - p_i)}{\|t_i - p_i\|_2} \right\|_2 = \sqrt{2} \sqrt{1 - \frac{\|P_{V_i^\perp}(t_i - p_i)\|_2}{\|t_i - p_i\|_2}}.$$

Hence (9) is at least

$$\begin{aligned} & \left\| \sum_{i=1}^n P_{V_i^\perp}(h) \right\|_2 - \|h\|_2 \sum_{i=1}^n \left\| \frac{P_{V_i^\perp}(t_i - p_i)}{\|P_{V_i^\perp}(t_i - p_i)\|_2} - \frac{(t_i - p_i)}{\|t_i - p_i\|_2} \right\|_2 - \frac{1}{\min_i(\|t_i - p_i\|_2^2)} \left\| \sum_{i=1}^n (t_i - p_i)(t_i - p_i)^* \right\|_{\text{op}} \|h\|_2 \\ &= \left\| \sum_{i=1}^n P_{V_i^\perp}(h) \right\|_2 - \|h\|_2 \sum_{i=1}^n \sqrt{2} \sqrt{1 - \frac{\|P_{V_i^\perp}(t_i - p_i)\|_2}{\|t_i - p_i\|_2}} - \frac{1}{\min_i(\|t_i - p_i\|_2^2)} \left\| \sum_{i=1}^n (t_i - p_i)(t_i - p_i)^* \right\|_{\text{op}} \|h\|_2. \end{aligned}$$

We will now expand the first term, $\|\sum_{i=1}^n P_{V_i^\perp}(h)\|_2$. Let $u = x - y$ and $z_i = x + y - (p_i + t_i)$. We have

$$\begin{aligned} P_{V_i^\perp}(h) &= P_{S(u, z_i)^\perp}(h) \\ &= P_{S(u, P_{u^\perp}(z_i))^\perp}(h) \\ &= h - P_u(h) - P_{P_{u^\perp}(z_i)}(h) \\ &= h - P_{P_{u^\perp}(z_i)}(h) + P_{z_i}(h) - P_{z_i}(h) \\ &= P_{z_i^\perp}(h) - P_{P_{u^\perp}(z_i)}(h) + P_{z_i}(h), \end{aligned}$$

where we used $h \perp u$ in the fourth inequality. Thus,

$$\begin{aligned} \left\| \sum_{i=1}^n P_{V_i^\perp}(h) \right\|_2 &= \left\| \sum_{i=1}^n \left(P_{z_i^\perp}(h) - P_{P_{u^\perp}(z_i)}(h) + P_{z_i}(h) \right) \right\|_2 \\ &\geq \left\| \sum_{i=1}^n P_{z_i^\perp}(h) \right\|_2 - \sum_{i=1}^n \|P_{P_{u^\perp}(z_i)}(h) - P_{z_i}(h)\|_2 \\ &\geq \left\| \sum_{i=1}^n P_{z_i^\perp}(h) \right\|_2 - \|h\|_2 \sum_{i=1}^n \left\| \frac{P_{u^\perp}(z_i)}{\|P_{u^\perp}(z_i)\|_2} - \frac{z_i}{\|z_i\|_2} \right\|_2 \\ &= \left\| \sum_{i=1}^n P_{z_i^\perp}(h) \right\|_2 - \|h\|_2 \sum_{i=1}^n \sqrt{2} \sqrt{1 - \frac{\|P_{u^\perp}(z_i)\|_2}{\|z_i\|_2}} \end{aligned}$$

Letting $X_i = \frac{\|P_{V^\perp}(t_i - p_i)\|_2}{\|t_i - p_i\|_2^2}$, $Y_i = \frac{\|P_{(x-y)^\perp}(x+y-t_i-p_i)\|_2}{\|x+y-t_i-p_i\|_2^2}$, and $Z_i = \|\sum_{i=1}^n (t_i - p_i)(t_i - p_i)^*\|_{\text{op}}$, we have shown that for any $h \perp x - y$,

$$\begin{aligned} \sum_{i=1}^n \|P_{W_i^\perp}(h)\|_2 &\geq \sum_{i=1}^n \left\| P_{(x+y-t_i-p_i)^\perp}(h) \right\|_2 \\ &\quad - \|h\|_2 \sum_{i=1}^n \sqrt{2} \left[\sqrt{1-X_i} + \sqrt{1-Y_i} \right] - \frac{1}{\min_i (\|t_i - p_i\|_2^2)} Z_i \|h\|_2 \end{aligned} \quad (10)$$

We will separately bound the first term and last two terms with high probability.

We now show that the first term of (10) is bounded below by $0.3n\|h\|_2$ with high probability.

Because $t_i + p_i =^d \sqrt{2}t_i$, it suffices to show that with high probability

$$\left\| \sum_{i=1}^n P_{(x+y-\sqrt{2}t_i)^\perp}(h) \right\|_2 \geq 0.3n\|h\|_2.$$

Let $v = x + y$ and $w_i = -\sqrt{2}t_i$. Note that

$$\begin{aligned} \left\| \sum_{i=1}^n P_{(v+w_i)^\perp}(h) \right\|_2 &= \left\| \sum_{i=1}^n \left(h - \frac{1}{\|v+w_i\|_2^2} \langle h, v+w_i \rangle (v+w_i) \right) \right\|_2 \\ &\geq n\|h\|_2 - \left\| \sum_{i=1}^n \frac{1}{\|v+w_i\|_2^2} (v+w_i)(v+w_i)^* h \right\|_2 \\ &\geq n\|h\|_2 - \left\| \sum_{i=1}^n \frac{1}{\|v+w_i\|_2^2} (v+w_i)(v+w_i)^* \right\|_{\text{op}} \|h\|_2 \\ &\geq \|h\|_2 \left[n - \frac{1}{\min_i \|v+w_i\|_2^2} \left\| \sum_{i=1}^n (v+w_i)(v+w_i)^* \right\|_{\text{op}} \right], \end{aligned}$$

where in the last inequality we used

$$\sum_{i=1}^n \frac{1}{\|v+w_i\|_2^2} (v+w_i)(v+w_i)^* \preceq \frac{1}{\min_i \|v+w_i\|_2^2} \sum_{i=1}^n (v+w_i)(v+w_i)^*.$$

Now, let $A = \sum_{i=1}^n e_i w_i^* \in \mathbb{R}^{n \times d}$. We have

$$\begin{aligned} \left\| \sum_{i=1}^n (v+w_i)(v+w_i)^* \right\|_{\text{op}} &= \left\| \sum_{i=1}^n (vv^* + vw_i^* + w_i v^* + w_i w_i^*) \right\|_{\text{op}} \\ &\leq n\|vv^*\|_{\text{op}} + \left\| v \left(\sum_{i=1}^n w_i \right)^* + \left(\sum_{i=1}^n w_i \right) v^* \right\|_{\text{op}} + \left\| \sum_{i=1}^n w_i w_i^* \right\|_{\text{op}} \\ &\leq n\|v\|_2^2 + 2\|v\|_2 \left\| \sum_{i=1}^n w_i \right\|_2 + \left\| \sum_{i=1}^n w_i w_i^* \right\|_{\text{op}} \\ &= n\|v\|_2^2 + 2\|v\|_2 \left\| \sum_{i=1}^n w_i \right\|_2 + \sigma_{\max}(A)^2. \end{aligned}$$

Thus,

$$\sum_{i=1}^n \left\| P_{(v+w_i)^\perp}(h) \right\|_2 \geq \|h\|_2 \left[n - \frac{n\|v\|_2^2 + 2\|v\|_2 \left\| \sum_{i=1}^n w_i \right\|_2 + \sigma_{\max}(A)^2}{\min_i \|v + w_i\|_2^2} \right].$$

Now, consider the event

$$E_1 = \left\{ \min_i \|v + w_i\|_2^2 \geq 4d\beta_1, \quad \|v\|_2^2 \leq 2d\beta_2, \quad \left\| \sum_{i=1}^n w_i \right\|_2^2 \leq 2nd\beta_3, \quad \sigma_{\max}(A)^2 \leq 2n\beta_4 \right\}$$

On E_1 we have,

$$\begin{aligned} \sum_{i=1}^n \left\| P_{(v+w_i)^\perp}(h) \right\|_2 &\geq \|h\|_2 \left[n - \frac{1}{4d\beta_1} \left(2nd\beta_2 + 2\sqrt{2d\beta_2}\sqrt{2}\sqrt{nd}\sqrt{\beta_3} + 2n\beta_4 \right) \right] \\ &= \|h\|_2 \left[n - \frac{1}{2}n\frac{\beta_2}{\beta_1} - \frac{4d\sqrt{n}\sqrt{\beta_2\beta_3}}{4d\beta_1} - \frac{\beta_4}{2d\beta_1}n \right] \\ &= \|h\|_2 \left[n \left(1 - \frac{1}{2}\frac{\beta_2}{\beta_1} - \frac{\beta_4}{2d\beta_1} - \frac{1}{\sqrt{n}}\frac{\sqrt{\beta_2\beta_3}}{\beta_1} \right) \right] \end{aligned}$$

Now, let $\beta_1 = 1 - \frac{1}{100}$, $\beta_2 = \beta_3 = 1 + \frac{1}{100}$, $\beta_4 = \frac{1}{5}d\beta_1$. This gives

$$\frac{1}{2}\frac{\beta_2}{\beta_1} = 1/2 + 1/99, \quad \frac{\beta_4}{2d\beta_1} = 1/10, \quad \frac{1}{\sqrt{n}}\frac{\sqrt{\beta_2\beta_3}}{\beta_1} < \frac{2}{\sqrt{n}}.$$

Assuming $n \geq 550$, we see that on E_1 ,

$$\sum_{i=1}^n \left\| P_{(x+y-t_i-p_i)^\perp}(h) \right\|_2 \geq 0.3n\|h\|_2.$$

Now, we bound $\mathbb{P}(E_1)$. Note that $\frac{1}{4}\|v + w_i\|_2^2 = \frac{d}{2}\|v\|_2^2 = \frac{d}{2n}\|\sum_{i=1}^n w_i\|_2^2 = \chi^2(d)$ and $\frac{1}{\sqrt{2}}A$ is a random $n \times d$ matrix with i.i.d. $\mathcal{N}(0, 1)$ entries. Thus, by applying Lemma 6, we have

$$\mathbb{P}\left(4d(1 - \epsilon) \leq \|v + w_i\|_2^2 \leq 4d(1 + \epsilon)\right) \geq 1 - e^{-c\epsilon^2 d}$$

$$\mathbb{P}\left(2d(1 - \epsilon) \leq \|v\|_2^2 \leq 2d(1 + \epsilon)\right) \geq 1 - e^{-c\epsilon^2 d}$$

$$\mathbb{P}\left(2nd(1 - \epsilon) \leq \left\| \sum_{i=1}^n w_i \right\|_2^2 \leq 2nd(1 + \epsilon)\right) \geq 1 - e^{-c\epsilon^2 d},$$

where $c > 0$ is a universal constant. Also by taking $t = 2\sqrt{d}$ in Lemma 7 we get

$$\mathbb{P}\left(\sigma_{\max}\left(\frac{1}{\sqrt{2}}A\right) \geq \sqrt{n} + 3\sqrt{d}\right) \leq 2e^{-2d}$$

We have

$$\mathbb{P}\left(\sigma_{\max}\left(\frac{1}{\sqrt{2}}A\right) \geq \sqrt{n\beta_4}\right) \leq \mathbb{P}\left(\sigma_{\max}\left(\frac{1}{\sqrt{2}}A\right) \geq \sqrt{n} + 3\sqrt{d}\right) \leq 2e^{-2d}$$

whenever $\sqrt{n} + 3\sqrt{d} \leq \sqrt{n\beta_4}$, or equivalently $(\sqrt{\frac{\beta_4}{5}}\sqrt{d} - 1)\sqrt{n} \geq 3\sqrt{d}$, which holds when $n \geq 550$ and $d \geq 10$. Thus for $n \geq 550$, we have

$$\mathbb{P}(E_1) \geq 1 - 2ne^{-cd}.$$

We now show that the second term of (10) is bounded above by $0.2n\|h\|_2$ with high probability. Define the event

$$E_2 = \left\{ X_i \geq 1 - \frac{1}{800}, \quad Y_i \geq 1 - \frac{1}{800}, \quad \frac{1}{\min_i(\|t_i - p_i\|_2^2)} Z_i \leq 0.1n, \quad i = 1, 2, \dots, n \right\}$$

On E_2 , we have

$$\|h\|_2 \sum_{i=1}^n \sqrt{2} \left[\sqrt{1 - X_i} + \sqrt{1 - Y_i} \right] + \frac{1}{\min_i(\|t_i - p_i\|_2^2)} Z_i \|h\|_2 \leq 0.2n\|h\|_2.$$

We now estimate $\mathbb{P}(E_2)$. For X_i , since $(t_i - p_i)$ is independent from $(x - y, p_i + t_i - x - y)$, we can view the latter as fixed. That is, by conditioning on V_i , and applying a rotation R such that $R(V_i) = \text{span}(e_1, e_2)$, we have

$$\frac{\|P_{V_i^\perp}(t_i - p_i)\|_2}{\|t_i - p_i\|_2} =_d \sqrt{\frac{\sum_{j=1}^{d-2} t_i(j)^2}{\sum_{j=1}^d t_i(j)^2}}$$

where $t_i(j)$ is the j th entry of t_i . As $\sum_{j=1}^{d-2} t_i(j)^2 \sim \chi_{d-2}^2$ and $\sum_{j=1}^d t_i(j)^2 \sim \chi_d^2$, Lemma 6 can be repeatedly applied to give $\mathbb{P}(X_i \geq 1 - \frac{1}{800} \text{ for all } i) \geq 1 - 2ne^{-cd}$. A similar argument gives $\mathbb{P}(Y_i \geq 1 - \frac{1}{800} \text{ for all } i) \geq 1 - 2ne^{-cd}$ because $x - y$ and $x + y - (t_i + p_i)$ are independent.

We now bound the probability of the third condition in the definition of E_2 . Note that

$$\frac{1}{\min_i(\|t_i - p_i\|_2^2)} \left\| \sum_{i=1}^n (t_i - p_i)(t_i - p_i)^* \right\|_{\text{op}} =_d \frac{1}{\min_i(\|t_i\|_2^2)} \left\| \sum_{i=1}^n t_i t_i^* \right\|_{\text{op}}$$

Let $B = \sum_{i=1}^n e_i t_i^*$. By Lemma 7, $\left\| \sum_{i=1}^n t_i t_i^* \right\|_{\text{op}} = \sigma_{\max}(B)^2 \geq n \left(1 + 3\sqrt{\frac{d}{n}} \right)^2$ with probability at least $1 - 2e^{-2d}$. By Lemma 6, $\|t_i\|_2^2 \geq d(1 - \varepsilon)$ for all i with probability at least $1 - ne^{-c\varepsilon^2 d}$. We conclude

$$\frac{1}{\min_i(\|t_i\|_2^2)} \left\| \sum_{i=1}^n t_i t_i^* \right\|_{\text{op}} \leq \frac{n \left(1 + 3\sqrt{\frac{d}{n}} \right)^2}{d(1 - \varepsilon)}$$

with probability at least $1 - 2ne^{-c\varepsilon^2 d}$. If $\varepsilon = 0.01, d \geq 40, n \geq 10d$, we have

$$\mathbb{P} \left(\frac{1}{\min_i(\|t_i - p_i\|_2^2)} Z_i \leq 0.1n \right) \geq 1 - 2ne^{-cd}$$

Hence, if $d \geq 40, n \geq 10d$,

$$\mathbb{P}(E_2) \geq 1 - 6ne^{-cd}.$$

In conclusion, there exist positive integers d_0 and n_0 such that for all $d \geq d_0, n \geq n_0, n \geq 10d$, and all $h \perp x - y$,

$$\mathbb{P} \left(\left\| \sum_{i=1}^n P_{W_i^\perp}(h) \right\|_2 \geq \frac{1}{10} \|h\|_2 \right) \geq 1 - \mathbb{P}[(E_1 \cap E_2)^c] \geq 1 - 6ne^{-cd}$$

for some $c > 0$, which implies the statement of the lemma. \square

2.7 Random graphs are p -typical with high probability

We prove that Condition 1 of Theorem 2 holds with high probability.

Lemma 14. *There exists an absolute constant $c > 0$ such that for all positive real numbers $p \leq 1$ satisfying $n_2 p \geq 2 \log(en_1)$ and $n_1 p \geq 2 \log(en_2)$, $G(n_1, n_2; p)$ is p -typical with probability at least $1 - n_1 n_2 2^{n_1+n_2} e^{-pn_1 n_2/4} - n_1^2 n_2 e^{-\Omega(n_2 p^2)} - n_1 n_2^2 e^{-\Omega(n_1 p^2)}$.*

Proof. Let V_1 and V_2 be vertex sets of sizes $|V_1| = n_1$ and $|V_2| = n_2$. Throughout the proof, we let $V_1 \cup V_2$ be the bipartition of the random graph $G(n_1, n_2; p)$. The bipartite graph $G(n_1, n_2; p)$ is not connected only if there exist partitions $V_1 = V_{1,1} \cup V_{1,2}$ and $V_2 = V_{2,1} \cup V_{2,2}$ such that the sets $V_{1,1} \cup V_{2,1}$ and $V_{1,2} \cup V_{2,2}$ are both non-empty and have no edges between them. Let $|V_{1,1}| = k_1, |V_{2,1}| = k_2, |V_{1,2}| = n_1 - k_1$ and $|V_{2,2}| = n_2 - k_2$. For fixed k_1, k_2 , by the union bound, the probability that there exists a partition as above is at most

$$\binom{n_1}{k_1} \binom{n_2}{k_2} (1-p)^{k_1(n_2-k_2)+k_2(n_1-k_1)}. \quad (11)$$

If $k_1 \leq \frac{n_1}{2}$ and $k_2 \leq \frac{n_2}{2}$, then by Stirling's formula, (11) is at most

$$\left(\frac{en_1}{k_1}\right)^{k_1} \left(\frac{en_2}{k_2}\right)^{k_2} (1-p)^{(k_1 n_2 + k_2 n_1)/2} \leq \left(\frac{en_1}{k_1} e^{-n_2 p/2}\right)^{k_1} \left(\frac{en_2}{k_2} e^{-n_1 p/2}\right)^{k_2}.$$

If $k_1 > \frac{n_1}{2}$ and $k_2 > \frac{n_2}{2}$, then let $\ell_1 = n_1 - k_1$ and $\ell_2 = n_2 - k_2$. Then (11) is at most

$$\left(\frac{en_1}{\ell_1}\right)^{\ell_1} \left(\frac{en_2}{\ell_2}\right)^{\ell_2} (1-p)^{(\ell_1 n_2 + \ell_2 n_1)/2} \leq \left(\frac{en_1}{\ell_1} e^{-n_2 p/2}\right)^{\ell_1} \left(\frac{en_2}{\ell_2} e^{-n_1 p/2}\right)^{\ell_2}.$$

If $(k_1 \leq \frac{n_1}{2} \text{ and } k_2 > \frac{n_2}{2})$ or $(k_1 > \frac{n_1}{2} \text{ and } k_2 \leq \frac{n_2}{2})$, then, by $\binom{n}{k} \leq 2^n$ for all $0 \leq k \leq n$, (11) is at most

$$2^{n_1+n_2} (1-p)^{n_1 n_2/4} \leq 2^{n_1+n_2} e^{-pn_1 n_2/4}.$$

Hence the probability that $G(n_1, n_2; p)$ is disconnected is at most

$$\begin{aligned} & \sum_{k_1=1}^{\lfloor n_1/2 \rfloor} \sum_{k_2=0}^{\lfloor n_2/2 \rfloor} \left(\frac{en_1}{k_1} e^{-n_2 p/2}\right)^{k_1} \left(\frac{en_2}{k_2} e^{-n_1 p/2}\right)^{k_2} + \sum_{k_1=0}^{\lfloor n_1/2 \rfloor} \sum_{k_2=1}^{\lfloor n_2/2 \rfloor} \left(\frac{en_1}{k_1} e^{-n_2 p/2}\right)^{k_1} \left(\frac{en_2}{k_2} e^{-n_1 p/2}\right)^{k_2} \\ & + \sum_{\ell_1=1}^{\lfloor n_1/2 \rfloor} \sum_{\ell_2=0}^{\lfloor n_2/2 \rfloor} \left(\frac{en_1}{\ell_1} e^{-n_2 p/2}\right)^{\ell_1} \left(\frac{en_2}{\ell_2} e^{-n_1 p/2}\right)^{\ell_2} + \sum_{\ell_1=0}^{\lfloor n_1/2 \rfloor} \sum_{\ell_2=1}^{\lfloor n_2/2 \rfloor} \left(\frac{en_1}{\ell_1} e^{-n_2 p/2}\right)^{\ell_1} \left(\frac{en_2}{\ell_2} e^{-n_1 p/2}\right)^{\ell_2} \\ & + n_1 n_2 2^{n_1+n_2} e^{-pn_1 n_2/4}, \end{aligned}$$

where the indeterminate factors in the sums corresponding to $k_1 = 0$, $k_2 = 0$, $\ell_1 = 0$, or $\ell_2 = 0$ are taken to be unity. Since $n_2 p \geq 2 \log(en_1)$ and $n_1 p \geq 2 \log(en_2)$, the four sums above are maximized at $(k_1, k_2) = (1, 0), (0, 1), (\ell_1, \ell_2) = (1, 0), (0, 1)$, respectively. Therefore the probability that $G(n_1, n_2; p)$ is disconnected is at most

$$2n_1 n_2 \cdot en_1 \cdot e^{-n_2 p/2} + 2n_1 n_2 \cdot en_2 \cdot e^{-n_1 p/2} + n_1 n_2 2^{n_1+n_2} e^{-pn_1 n_2/4}.$$

For a fixed vertex $v \in V_1$, the expected value of $\deg(v)$ is $n_2 p$, and for a pair of vertices $v, w \in V_1$, the expected value of the codegree of v and w is $n_2 p^2$. Therefore by Chernoff's inequality (see Fact 4 from [1]) and a union bound, the probability that all vertices in V_1 have degree between $\frac{1}{2}n_2 p$ and $2n_2 p$, and all pairs of vertices in V_1 have codegree between $\frac{1}{2}n_2 p^2$ and $2n_2 p^2$ is $1 - n_1^2 e^{-\Omega(n_2 p^2)}$. Similarly, the probability that all vertices in V_2 have degree between $\frac{1}{2}n_1 p$ and $2n_1 p$, and all pairs of vertices in V_2 have codegree between $\frac{1}{2}n_1 p^2$ and $2n_1 p^2$ is $1 - n_2^2 e^{-\Omega(n_1 p^2)}$. The conclusion follows by taking a union bound over all events. \square

2.8 Proof of Theorem 1

We can now prove the high dimensional recovery theorem, which we state here again for convenience:

Theorem 1. *Let $N = \max(n_\ell, n_s)$, $n = \min(n_\ell, n_s)$. Let $G(V_\ell \cup V_s, E)$ be drawn from a bipartite-Erdős-Rényi graph with $p > 0$. Take $t_1^{(0)}, \dots, t_{n_\ell}^{(0)}, p_1^{(0)}, \dots, p_{n_s}^{(0)} \sim \mathcal{N}(0, I_{d \times d})$ to be independent from each other and G . Then, there exist absolute constants $c, c_3, C > 0$ such that for $\gamma = c_3 p^4$, if*

$$\max\left(\frac{1}{c_3 p^4}, Cd, \frac{2 \log(eN)}{p}, \Omega(c_3 \log^2 N)\right) \leq n \leq N \leq e^{\frac{1}{8}cd}$$

and $d = \Omega(1)$, then there exists an event with probability at least $1 - O(e^{-\Omega(\frac{1}{2}c_3^{-1/2}n^{1/2})} + e^{-\frac{1}{2}cd})$, on which the following holds:

For all subgraphs E_b satisfying $\max_{i \in [n_\ell]} \deg_b(t_i) \leq \gamma n_s$ and $\max_{j \in [n_s]} \deg_b(p_j) \leq \gamma n_\ell$ and all pairwise direction corruptions $v_{ij} \in \mathbb{S}^{d-1}$ for $ij \in E_b$, the convex program (2) has a unique minimizer equal to $\left\{ \alpha \{t_i^{(0)} - \zeta\}_{i \in [n_\ell]}, \alpha \{p_j^{(0)} - \zeta\}_{j \in [n_s]} \right\}$ for some positive α and for $\zeta = \frac{1}{n_\ell + n_s} \left(\sum_{i \in [n_\ell]} t_i^{(0)} + \sum_{j \in [n_s]} p_j^{(0)} \right)$.

Proof. Let c be minimum of the constants from Lemmas 10 and 11. Let K_0 be the constant from Lemma 11. It is enough to verify that G , T and E_b in the assumption of the present theorem satisfy the deterministic conditions 1–6 in Theorem 2, with appropriate constants $p, \beta, c_0, \epsilon, c_1$, and with the purported probability. By Lemma 14, Condition 1 holds with probability at least

$$1 - n_\ell n_s 2^{n_\ell + n_s} e^{-pn_\ell n_s/4} - n_\ell^2 n_s e^{-\Omega(n_s p^2)} - n_\ell n_s^2 e^{-\Omega(n_\ell p^2)} = 1 - O(N^3 e^{-\Omega(np^2)})$$

if $np \geq 2 \log(eN)$. Condition 2 holds with probability 1. By Lemma 10, Condition 3 holds for $c_0 = \frac{9}{10}$ with probability at least $1 - 2n_\ell n_s e^{-cd}$, and Condition 4 holds for $\beta = \frac{1}{4}$ with probability at least $1 - 22n_\ell^2 n_s^2 e^{-cd}$. By Lemma 11, Condition 5 holds for $c_1 = \frac{1}{20}$ with probability $1 - O(n_\ell^2 n_s^2 e^{-cd})$ if $n \geq \max(K_0, 160d)$, and $d \geq d_0$. Thus, Conditions 1–5 hold together with probability at least

$$1 - O(N^4 e^{-cd} + N^3 e^{-\Omega(np^2)}).$$

Take $\gamma = c_3 p^4 \leq \frac{p^4}{10^{11}}$. Because $\gamma \leq \frac{\beta c_0 c_1^2 p^4}{384 \cdot 204 \cdot 64}$, Theorem 2 implies that recovery via ShapeFit is guaranteed. Note that the conditions $\max_{i \in V_\ell} \deg_b(i) \leq \gamma n_s$ and $\max_{j \in V_s} \deg_b(j) \leq \gamma n_\ell$ are nontrivial when $p \geq c_3^{-1/4} n^{-1/4}$. Using this inequality, we have $N^3 e^{-\Omega(np^2)} \leq N^3 e^{-\Omega(c_3^{-1/2} n^{1/2})} \leq e^{-\Omega(\frac{1}{2}c_3^{-1/2} n^{1/2})}$ if $n = \Omega(c_3 \log^2 N)$ and $N^4 e^{-cd} \leq e^{-cd/2}$ if $N \leq e^{\frac{1}{8}cd}$. Thus, the probability of exact recovery via ShapeFit, uniformly in E_b and v_{ij} satisfying the assumptions of the theorem, is at least

$$1 - O(e^{-\Omega(\frac{1}{2}c_3^{-1/2} n^{1/2})} + e^{-\frac{1}{2}cd}). \quad \square$$

3 Numerical simulations

In this section, we use numerical simulation to verify that ShapeFit recovers Gaussian camera locations and Gaussian structure locations in \mathbb{R}^3 in the presence of corrupted pairwise direction measurements. Further, we empirically demonstrate that ShapeFit is robust to noise in the uncorrupted measurements.

Let $\tilde{t}_i^{(0)} \in \mathbb{R}^3$ be independent $\mathcal{N}(0, I_{3 \times 3})$ random variables for $i = 1, \dots, n_\ell$. Let $\tilde{p}_j^{(0)} \in \mathbb{R}^3$ be independent $\mathcal{N}(0, I_{3 \times 3})$ random variables for $j = 1, \dots, n_s$. Let

$$t_i^{(0)} = \tilde{t}_i^{(0)} - \frac{1}{n_\ell + n_s} \left(\sum_k \tilde{t}_k^{(0)} + \sum_\ell \tilde{p}_\ell^{(0)} \right) \text{ and } p_j^{(0)} = \tilde{p}_j^{(0)} - \frac{1}{n_\ell + n_s} \left(\sum_k \tilde{t}_k^{(0)} + \sum_\ell \tilde{p}_\ell^{(0)} \right).$$

Let the graph of observations G be a bipartite Erdős-Rényi graph $G(n_\ell, n_s, p)$ on $n_\ell + n_s$ vertices, for $p = 1/2$. For $ij \in E(G)$, let

$$\tilde{v}_{ij} = \begin{cases} z_{ij} & \text{with probability } q, \\ \frac{t_i^{(0)} - p_j^{(0)}}{\|t_i^{(0)} - p_j^{(0)}\|_2} + \sigma z_{ij} & \text{with probability } 1 - q, \end{cases}$$

where z_{ij} are independent and uniform over \mathbb{S}^2 . Let $v_{ij} = \tilde{v}_{ij} / \|\tilde{v}_{ij}\|_2$. That is, each observation is corrupted with probability q , and each corruption is in a random direction. In the noiseless case, with $\sigma = 0$, each observation is exact with probability $1 - q$.

We solved ShapeFit using the SDPT3 solver [8, 9] and YALMIP [5]. For output $S = (T, P) = (\{t_i\}_{i \in [n_\ell]}, \{p_j\}_{j \in [n_s]})$, define its relative error with respect to $S^{(0)} = (T^{(0)}, P^{(0)}) = (\{t_i^{(0)}\}_{i \in [n_\ell]}, \{p_j^{(0)}\}_{j \in [n_s]})$ as

$$\left\| \frac{S}{\|S\|_F} - \frac{S^{(0)}}{\|S^{(0)}\|_F} \right\|_F$$

where $\|S\|_F$ is the Frobenius norm of the matrix whose column are given by $\{t_i\}$ and $\{p_j\}$. This error metric amounts to an ℓ_2 norm after rescaling.

Figure 1 shows the mean relative error of the output of ShapeFit over 10 independent trials for locations in \mathbb{R}^3 generated by $p = 1/2$, $n_\ell = n_s$, $\sigma \in [0, 0.05]$, and a range of values $10 \leq n_\ell + n_s \leq 70$ and $0 \leq q \leq 0.5$. White blocks represent zero average relative error, and black blocks represent an average relative error of 1 or higher. Average residuals between 0 and 1 are represented by the appropriate shade of gray. The figure shows that ShapeFit can empirically recover 3d locations in the presence of a surprisingly large probability of corruption, provided n is big enough. For example, if $n \geq 50$, ShapeFit outputs a structure with small relative error even when around 15% of all measurements are randomly corrupted. Further, successful recovery occurs both in the noiseless case, and in the noisy case with $\sigma = 0.05$.

Figure 2 shows the median residual over 10 independent trials for locations in \mathbb{R}^3 generated by $p = 1/2$, $n_\ell = n_s = 25$, $q = 0.1$ and a range of values of $10^{-6} \leq \sigma \leq 10^0$. We see that ShapeFit is empirically stable to noise, with median residuals that are approximately linear in the noise parameter σ .

Acknowledgements

We are grateful to Stefano Soatto for suggesting to VV the problem formulation addressed in this paper. VV is partially supported by the Office of Naval Research. CL is partially supported by the National Science Foundation Grant DMS-1362326. PH is partially supported by the National Science Foundation Grant DMS-1418971.

References

- [1] Dana Angluin and Leslie G. Valiant. Fast probabilistic algorithms for hamiltonian circuits and matchings. In *Proceedings of the Ninth Annual ACM Symposium on Theory of Computing*, STOC '77, pages 30–41, New York, NY, USA, 1977. ACM.
- [2] Reinhard Diestel. *Graph theory*. Springer-Verlag Berlin and Heidelberg, third edition, 2005.

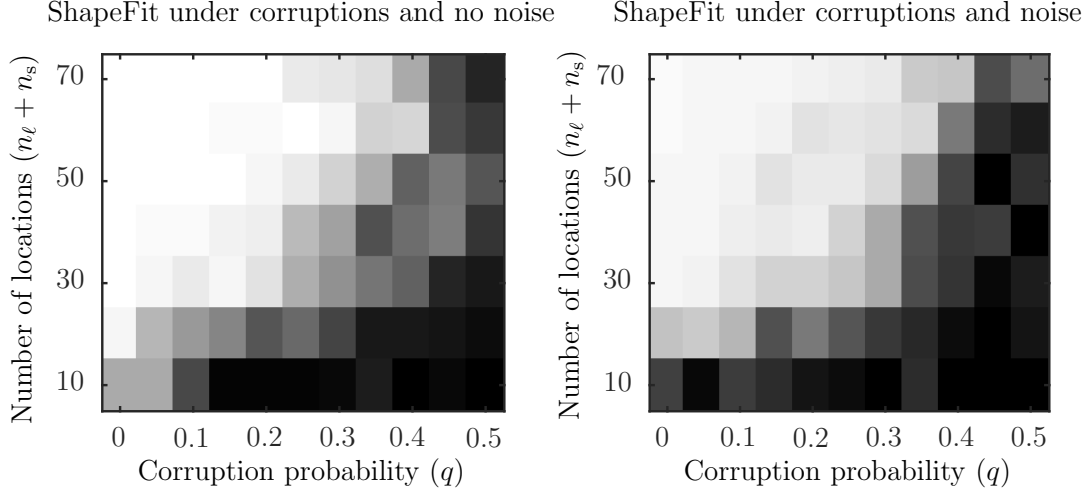


Figure 1: Mean recovery error of ShapeFit as a function of the number of locations $n_\ell + n_s$ and the corruption probability q . The data model has 3d Gaussian locations whose pairwise directions are observed in accordance with a bipartite Erdős-Rényi graph $G(n_\ell, n_s, 1/2)$ and are corrupted with probability q . White blocks represent an average relative error of zero over 10 independently generated problems. Black blocks represent an average relative error of 100%. The left panel corresponds to the noiseless case $\sigma = 0$, and the right panel corresponds to the noisy case $\sigma = 0.05$.

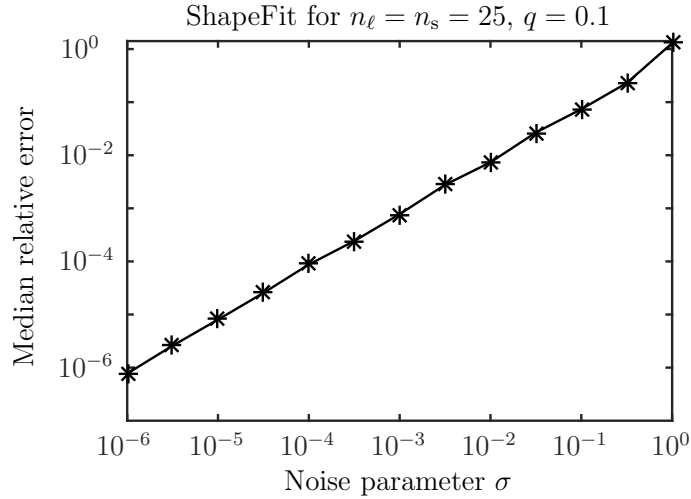


Figure 2: Median recovery error of ShapeFit versus the noise parameter σ . These simulations are based on 50 Gaussian locations in \mathbb{R}^3 whose pairwise directions are observed in accordance with a bipartite Erdős-Rényi graph $G(25, 25, 1/2)$ and are corrupted with probability $q = 0.1$. The median is based on 10 independently generated problems.

- [3] Venu Madhav Govindu. Lie-algebraic averaging for globally consistent motion estimation. In *Computer Vision and Pattern Recognition, 2004. CVPR 2004. Proceedings of the 2004 IEEE Computer Society Conference on*, volume 1, pages I–684. IEEE, 2004.
- [4] Paul Hand, Choongbum Lee, and Vladislav Voroninski. Shapefit: Exact location recovery from corrupted pairwise directions. *CoRR*, abs/1506.01437, 2015.
- [5] J. Löfberg. Yalmip : A toolbox for modeling and optimization in MATLAB. In *Proceedings of the CACSD Conference*, Taipei, Taiwan, 2004.
- [6] Yi Ma, Stefano Soatto, Jana Kosecka, and S. Shankar Sastry. *An Invitation to 3-D Vision: From Images to Geometric Models*. SpringerVerlag, 2003.
- [7] Onur Özyeşil and Amit Singer. Robust camera location estimation by convex programming. *Proceedings of Computer Vision and Pattern Recognition.*, 2015.
- [8] K. C. Toh, M.J. Todd, and R. H. Tutuncu. Sdpt3 - a matlab software package for semidefinite programming. *Optimization Methods and Software*, 11:545–581, 1998.
- [9] R.H. Tutuncu, K.C. Toh, and M.J. Todd. Solving semidefinite-quadratic-linear programs using sdpt3. *Mathematical Programming Ser. B*, 95:189–217, 2003.
- [10] R. Vershynin. Introduction to the non-asymptotic analysis of random matrices. In Y.C. Eldar and G. Kutyniok, editors, *Compressed Sensing: Theory and Applications*. Cambridge University Press, 2012.
- [11] Lanhui Wang and Amit Singer. Exact and stable recovery of rotations for robust synchronization. *Information and Inference*, 2013.
- [12] Kyle Wilson and Noah Snavely. Robust global translations with 1dsfm. In *Proceedings of the European Conference on Computer Vision (ECCV)*, 2014.